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A note on an integral of Dixit, Roy and Zaharescu

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Abstract

In a recent paper, Dixit *et al.* [Acta Arith. **177** (2017) 1–37] posed two open questions whether the integral

$$\hat{J}_k(\alpha) = \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,{}_1F_1(-k, \tfrac{3}{2}; 2\alpha x^2)\,dx$$

for $\alpha > 0$ could be evaluated in closed form when k is a positive even and odd integer. We establish that $\hat{J}_k(\alpha)$ can be expressed in terms of a Gauss hypergeometric function and a ratio of two gamma functions, together with a remainder expressed as an integral. An upper bound on the remainder term is obtained, which is shown to be exponentially small as k becomes large when $a = O(1)$.

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1. Introduction In the first of his letters to Hardy [5], Ramanujan gave the formula

$$I(\alpha) := \alpha^{-1/4}\left(1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right) = \beta^{-1/4}\left(1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right),$$

where $\alpha\beta = \pi^2$, and in [6] obtained the approximate evaluation

$$I(\alpha) \simeq \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4}. \tag{1.1}$$

This approximation is found to be good for small and large values of α. A proof of this result was given in [1], where the asymptotic expansion

$$I(\alpha) \sim \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \frac{\alpha^{7/4}}{60} + \cdots \qquad (\alpha \to 0)$$

was obtained.

In a recent paper, Dixit, Roy and Zaharescu [2] established an analogous formula for the integral

$$\hat{J}_k(\alpha) := \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,{}_1F_1(-k;\tfrac{3}{2};2\alpha x^2)\,dx,$$

where ${}_1F_1$ denotes the confluent hypergeometric function and k is a positive integer. They showed that [2, (1.25), (1.27)]

$$\alpha^{-1/4}{}_2F_1(-2k,1;\tfrac{3}{2};2)+4\alpha^{3/4}\hat{J}_{2k}(\alpha)=\beta^{-1/4}{}_2F_1(-2k,1;\tfrac{3}{2};2)+4\beta^{3/4}\hat{J}_{2k}(\beta)$$

and

$$\alpha^{-1/4}{}_2F_1(-2k-1,1;\tfrac{3}{2};2)+4\alpha^{3/4}\hat{J}_{2k+1}(\alpha)$$
$$=-\beta^{-1/4}{}_2F_1(-2k-1,1;\tfrac{3}{2};2)-4\beta^{3/4}\hat{J}_{2k+1}(\beta) \qquad (1.2)$$

when $\alpha\beta=\pi^2$, where ${}_2F_1$ denotes the Gauss hypergeometric function. In the particular case $\alpha=\beta=\pi$, (1.2) yields the beautiful exact evaluation [2, Cor. 1.8]

$$\hat{J}_{2k+1}(\pi) := \int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x}-1}\,{}_1F_1(-2k-1;\tfrac{3}{2};2\pi x^2)\,dx=-\frac{1}{4\pi}\,{}_2F_1(-2k-1,1;\tfrac{3}{2};2) \qquad (1.3)$$

for $k=0,1,2,\ldots$. In addition, they gave the approximation [2, (1.26)]

$$4\alpha^{3/4}\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,{}_1F_1(-2k;\tfrac{3}{2};2\alpha x^2)\,dx$$
$$\simeq {}_2F_1(-2k,1;\tfrac{3}{2};2)\Big\{-\alpha^{-1/4}+\Big(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{3\cdot{}_2F_1(-2k,1;\frac{3}{2};2)}\Big)^{1/4}\Big\}, \qquad (1.4)$$

which reduces to (1.1) when $k=0$.

At the end of their paper, Dixit *et al.* posed the following two open questions, namely:

Question 1. Find the exact evaluation of the integral

$$\hat{J}_{2k}(\pi) := \int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x}-1}\,{}_1F_1(-2k;\tfrac{3}{2};2\pi x^2)\,dx \qquad (1.5)$$

for positive integer k.

Question 2. Find the exact evaluation of, or at least an approximation to, the integral

$$\hat{J}_{2k+1}(\alpha) = \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,{}_1F_1(-2k-1;\tfrac{3}{2};2\alpha x^2)\,dx \qquad (1.6)$$

when $\alpha \neq \pi$ is a positive real number and k is a non-negative integer. In this note we partially answer the above two questions by obtaining simple closed-form expressions for these integrals which, although not exact, approximate the given integrals to within exponentially small accuracy when k is large and $a = O(1)$. In addition, we extend the scope of Question 1 by considering the integral $\hat{J}_{2k}(\alpha)$ with $\alpha > 0$ and, as a by-product of the analysis pertaining to Question 2, we supply an alternative proof of the result (1.3).

2. The analysis of $J_{2k}(a)$ Throughout we shall find it convenient to replace the parameter α by πa and define the integral $J_{2k}(a)$ by

$$J_{2k}(a) = \int_0^\infty \frac{xe^{-\pi a x^2}}{e^{2\pi x}-1}\, {}_1F_1(-2k; \tfrac{3}{2}; 2\pi a x^2)\, dx \tag{2.1}$$

for $a > 0$ and positive integer k. Then $J_{2k}(1) = \hat{J}_{2k}(\pi)$ in (1.5). The confluent hypergeometric function terminates and we have [4, p. 322]

$${}_1F_1(-2k; \tfrac{3}{2}; 2\pi x^2) = \sum_{r=0}^{2k} \frac{(-2k)_r}{(\frac{3}{2})_r r!}(2\pi a x^2)^r.$$

Substitution of this series into the left-hand side of the above yields

$$J_{2k}(a) = \sum_{r=0}^{2k} \frac{(-2k)_r}{(\frac{3}{2})_r r!}(2\pi a)^r \int_0^\infty \frac{x^{2r+1}e^{-\pi a x^2}}{e^{2\pi x}-1}\, dx$$

upon reversal of the order of summation and integration.

Now

$$\int_0^\infty \frac{x^{2r+1}e^{-\pi a x^2}}{e^{2\pi x}-1}\, dx = \sum_{n\geq 1} \int_0^\infty x^{2r+1} e^{-\pi a x^2 - 2\pi n x} dx = \frac{a^{1/2} r!\Gamma(r+\frac{3}{2})}{2(\pi a)^{r+3/2}}\, U_r,$$

where

$$U_r := \sum_{n\geq 1} U(r+1, \tfrac{1}{2}, \pi n^2/a)$$

with $U(a, b, z)$ being the confluent hypergeometric function of the second kind [4, p. 322]. Then we obtain

$$J_{2k}(a) = \frac{1}{4\pi a} \sum_{r=0}^{2k} (-2k)_r 2^r U_r.$$

From the integral representation [4, p. 326]

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1} dt \qquad (a > 0,\, \Re(z) > 0), \tag{2.2}$$

we find

$$U_r = \sum_{n\geq 1} \frac{1}{r!} \int_0^\infty e^{-\pi n^2 t/a} t^r (1+t)^{-r-3/2} dt = \frac{1}{r!} \int_0^\infty \frac{\psi(t)\, t^r}{(1+t)^{r+3/2}}\, dt,$$

where we have defined

$$\psi(t) := \sum_{n\geq 1} e^{-\pi n^2 t/a}. \tag{2.3}$$

Hence

$$\begin{aligned} J_{2k}(a) &= \frac{1}{4\pi a} \int_0^\infty \sum_{r=0}^{2k} \frac{(-2k)_r}{r!} \left(\frac{2t}{1+t}\right)^r \frac{\psi(t)}{(1+t)^{3/2}}\, dt \\ &= \frac{1}{4\pi a} \int_0^\infty \frac{\psi(t)(1-t)^{2k}}{(1+t)^{2k+3/2}}\, dt, \end{aligned} \tag{2.4}$$

where the finite sum has been evaluated as [4, (15.4.6)]

$${}_1F_0\Big(-2k;\,;\frac{2t}{1+t}\Big) = \Big(\frac{1-t}{1+t}\Big)^{2k}.$$

We now divide the integration path into $[0,1]$ and $[1,\infty)$ and make the change of variable $t \to 1/t$ in the integral over $[0,1]$. This yields

$$J_{2k}(a) = \frac{1}{4\pi a} \int_1^\infty \{t^{-1/2}\psi(1/t) + \psi(t)\}\, \frac{(t-1)^{2k}}{(1+t)^{2k+3/2}}\, dt.$$

For the sum

$$\Psi(\tau) = \sum_{n\geq 1} e^{-\pi n^2 \tau}, \tag{2.5}$$

we have the well-known Poisson transformation given by [7, p. 124]

$$\Psi(\tau) + \tfrac{1}{2}(1 - \tau^{-1/2}) = \tau^{-1/2}\Psi(1/\tau). \tag{2.6}$$

With $\tau = at$, this yields

$$t^{-1/2}\psi(1/t) = a^{1/2}\{\phi(t) + \tfrac{1}{2}(1-(at)^{-1/2})\}, \qquad \phi(t) := \sum_{n\geq 1} e^{-\pi n^2 at}. \tag{2.7}$$

Hence

$$\begin{aligned} J_{2k}(a) &= \frac{1}{4\pi a} \int_1^\infty \Big\{\psi(t) + a^{1/2}\phi(t) + \tfrac{1}{2}a^{1/2}(1-(at)^{-1/2})\Big\}\, \frac{(t-1)^{2k}}{(1+t)^{2k+3/2}}\, dt \\ &= -\frac{1}{8\pi a} \int_0^1 (1 - a^{1/2}t^{-1/2})\, \frac{(1-t)^{2k}}{(1+t)^{2k+3/2}}\, dt \\ &\qquad + \frac{1}{4\pi a} \int_1^\infty \Big\{\psi(t) + a^{1/2}\phi(t)\Big\}\, \frac{(t-1)^{2k}}{(1+t)^{2k+3/2}}\, dt. \end{aligned}$$

For positive integer k, we have the integrals

$$\int_0^1 \frac{t^{-1/2}(1-t)^{2k}}{(1+t)^{2k+3/2}}\,dt = \sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+1)}{\Gamma(2k+\frac{3}{2})}$$

and

$$\begin{aligned}\int_0^1 \frac{(1-t)^{2k}}{(1+t)^{2k+3/2}}\,dt &= \frac{1}{2k+1}\,{}_2F_1(1,2k+\tfrac{3}{2};2k+2;-1)\\ &= 2\,{}_2F_1(-2k,1;\tfrac{3}{2};2)-\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+1)}{\Gamma(2k+\frac{3}{2})}\end{aligned}$$

by application of the transformation [4, p. 390]

$${}_2F_1(a,b;c;z)=\frac{\Gamma(a)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,z^{-a}{}_2F_1(a,a-c+1;a+b-c+1;1-z^{-1})$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,z^{a-c}(1-z)^{c-a-b}{}_2F_1(c-a,1-a;c-a-b+1;1-z^{-1}). \quad (2.8)$$

Hence we obtain

Theorem 1. *Let $a>0$ and k be a positive integer. Then the integral $J_{2k}(a)$ defined in (2.1) satisfies*

$$J_{2k}(a)=T_{2k}(a)+\epsilon_{2k}(a), \quad (2.9)$$

where

$$T_{2k}(a)=\frac{1}{4\pi a}\Big\{\Big(\frac{1+a^{1/2}}{2}\Big)\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+1)}{\Gamma(2k+\frac{3}{2})}-{}_2F_1(-2k,1;\tfrac{3}{2};2)\Big\} \quad (2.10)$$

and

$$\epsilon_{2k}(a)=\frac{1}{4\pi a}\int_1^\infty \{\psi(t)+a^{1/2}\phi(t)\}\frac{(t-1)^{2k}}{(1+t)^{2k+3/2}}\,dt \quad (2.11)$$

with the sums $\psi(t)$ and $\phi(t)$ defined in (2.3) and (2.7).

It will be found subsequently that $\epsilon_{2k}(a)$ is small for $k\geq 1$ when $a=O(1)$ and so we shall refer to it as the remainder term. We observe that when $a=1$, we have $\phi(t)=\psi(t)$ and hence that

$$\epsilon_{2k}(1)=\frac{1}{2\pi}\int_1^\infty \frac{\psi(t)(t-1)^{2k}}{(1+t)^{2k+3/2}}\,dt.$$

3. The analysis of $J_{2k+1}(a)$ A similar treatment for the integral

$$J_{2k+1}(a)=\int_0^\infty \frac{xe^{-\pi ax^2}}{e^{2\pi x}-1}\,{}_1F_1(-2k-1;\tfrac{3}{2};2\pi ax^2)\,dx \qquad (k=0,1,2,\ldots) \quad (3.1)$$

shows that

$$J_{2k+1}(a)=\frac{1}{4\pi a}\sum_{r=0}^{2k+1}(-2k-1)_r 2^r U_r=\frac{1}{4\pi a}\int_0^\infty \frac{\psi(t)\,(1-t)^{2k+1}}{(1+t)^{2k+5/2}}\,dt.$$

Dividing the integration path as in Section 2, we find

$$J_{2k+1}(a)=\frac{1}{4\pi a}\int_1^\infty \{t^{-1/2}\psi(1/t)-\psi(t)\}\,\frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}}\,dt.$$

Application of (2.7) and some straightforward algebra then produces

$$J_{2k+1}(a)=-\frac{1}{8\pi a}\int_0^1 (1-a^{1/2}t^{-1/2})\,\frac{(1-t)^{2k+1}}{(1+t)^{2k+5/2}}\,dt$$
$$+\frac{1}{4\pi a}\int_1^\infty \{a^{1/2}\phi(t)-\psi(t)\}\,\frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}}\,dt.$$

Now

$$\int_0^1 (1-a^{1/2}t^{-1/2})\,\frac{(1-t)^{2k+1}}{(1+t)^{2k+5/2}}\,dt$$
$$=\frac{1}{2k+2}\,{}_2F_1(1,2k+\tfrac{5}{2};2k+3;-1)-\sqrt{\frac{\pi a}{2}}\,\frac{\Gamma(2k+2)}{\Gamma(2k+\frac{5}{2})}$$
$$=2\,{}_2F_1(-2k-1,1;\tfrac{3}{2};2)+(1-a^{1/2})\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+2)}{\Gamma(2k+\frac{5}{2})}$$

by (2.8). Hence we obtain

Theorem 2. *Let $a>0$ and k be a non-negative integer. Then the integral $J_{2k+1}(a)$ defined in (3.1) satisfies*

$$J_{2k+1}(a)=-T_{2k+1}(a)+\epsilon_{2k+1}(a), \quad (3.2)$$

where

$$T_{2k+1}(a)=\frac{1}{4\pi a}\left\{\left(\frac{1-a^{1/2}}{2}\right)\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+2)}{\Gamma(2k+\frac{5}{2})}+{}_2F_1(-2k-1,1;\tfrac{3}{2};2)\right\} \quad (3.3)$$

and

$$\epsilon_{2k+1}(a)=\frac{1}{4\pi a}\int_1^\infty \{a^{1/2}\phi(t)-\psi(t)\}\frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}}\,dt \quad (3.4)$$

with the sums $\psi(t)$ and $\phi(t)$ defined in (2.3) and (2.7).

When $a=1$, we have $\psi(t)=\phi(t)$ and hence $\epsilon_{2k+1}(1)=0$. It then follows from (3.2) that

$$J_{2k+1}(1)=-\frac{1}{4\pi}\,{}_2F_1(-2k-1,1;\tfrac{3}{2};2),$$

which supplies another proof of the result stated in (1.3) obtained in [2].

4. Estimation of the remainder terms We examine the remainder terms $\epsilon_{2k}(a)$ and $\epsilon_{2k+1}(a)$ appearing in (2.11) and (3.4) and determine bounds and an estimate of their large-k behaviour. We consider first the term $\epsilon_{2k}(a)$ which can be written as

$$\epsilon_{2k}(a)=\frac{a^{-3/4}}{4\pi}\int_1^\infty \{a^{1/4}\phi(t)+a^{-1/4}\psi(t)\}\,\frac{(t-1)^{2k}}{(1+t)^{2k+3/2}}\,dt$$

With the change of variable $t\to 1+u$, we have

$$\epsilon_{2k}(a)=\frac{a^{-3/4}}{4\pi}\biggl\{a^{1/4}\sum_{n\geq 1}e^{-\pi n^2a}\int_0^\infty e^{-\pi n^2au}h(u)\,du$$

$$+a^{-1/4}\sum_{n\geq 1}e^{-\pi n^2/a}\int_0^\infty e^{-\pi n^2u/a}h(u)\,du\biggr\}$$

$$<\frac{a^{-3/4}}{4\pi}\biggl\{a^{1/4}\Psi(a)\int_0^\infty e^{-\pi au}h(u)\,du+a^{-1/4}\Psi(1/a)\int_0^\infty e^{-\pi u/a}h(u)\,du\biggr\}, \qquad (4.1)$$

where $\Psi(a)$ is defined in (2.5) and $h(u)=u^{2k}/(2+u)^{2k+3/2}$. Evaluation of the integrals appearing in (4.1) in terms of the confluent hypergeometric function $U(a,b,z)$ by (2.2), we then obtain the upper bound in the form

Theorem 3. *The remainder term $\epsilon_{2k}(a)$ defined in (2.11) satisfies the upper bound*

$$\epsilon_{2k}(a)<\mathcal{B}_{2k}(a), \qquad \mathcal{B}_{2k}(a):=\frac{a^{-3/4}(2k)!}{4\sqrt{2\pi}}\{E_{2k}(a)+E_{2k}(1/a)\}, \qquad (4.2)$$

where

$$E_{2k}(a):=a^{1/4}\Psi(a)U(2k+1,\tfrac{1}{2},2\pi a).$$

and $\Psi(a)$ is given by (2.5).

The behaviour of this bound as $k\to\infty$ with a fixed can be obtained by making use of the result [4, (13.8.8)]

$$U(2k+1,\tfrac{1}{2},2\pi a)\sim\frac{e^{\pi a}}{(2k)!}\sqrt{\frac{\pi}{2k}}\,e^{-4\sqrt{\pi ak}} \qquad (k\to\infty,\ a\ll 2k/\pi).$$

For values of $a \simeq 1$, we can bound the sum $\Psi(a)$ by

$$\Psi(a) := \sum_{n\geq 1} e^{-\pi n^2 a} = e^{-\pi a}\Big(1 + e^{-3\pi a} + e^{\pi a}\sum_{n\geq 3} e^{-\pi n^2 a}\Big) < \lambda(a)e^{-\pi a},$$

where

$$\lambda(a) := 1 + e^{-3\pi a} + e^{\pi a}\sum_{n\geq 3} e^{-\pi n a} = 1 + e^{-3\pi a} + \frac{e^{-2\pi a}}{1 - e^{-\pi a}}. \qquad (4.3)$$

This then yields the estimate as $k \to \infty$

$$\mathcal{B}_{2k}(a) \sim \frac{a^{-3/4}k^{-1/2}}{8\sqrt{\pi}}\Big\{a^{1/4}\lambda(a)e^{-4\sqrt{\pi a k}} + a^{-1/4}\lambda(1/a)e^{-4\sqrt{\pi k/a}}\Big\} \qquad (4.4)$$

provided $a \gg \pi/(2k)$ and $a \ll 2k/\pi$ (that is, when a is neither too small nor too large). In the case $a = 1$ we have

$$\mathcal{B}_{2k}(1) \sim \frac{\lambda(1)}{4\sqrt{\pi}}\, k^{-1/2}e^{-4\sqrt{\pi k}} \qquad (k \to \infty).$$

The remainder term $\epsilon_{2k+1}(a)$ may be written as

$$\epsilon_{2k+1}(a) = \frac{a^{-3/4}}{4\pi}\int_1^\infty \{a^{1/4}\phi(t) - a^{-1/4}\psi(t)\}\,\frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}}\,dt.$$

It is straightforward to show (we omit these details) that $a^{1/2}\phi(t) - \psi(t)$ has opposite signs in the intervals $a \in (0,1)$ and $a \in (1,\infty)$ when $t \in [1,\infty)$, being negative in $a \in (1,\infty)$. Hence it follows that $\epsilon_{2k+1}(a) < 0$ when $a \in (1,\infty)$ and $\epsilon_{2k+1}(a) > 0$ when $a \in (0,1)$. The same procedure employed for $\epsilon_{2k}(a)$ shows that[1]

$$|\epsilon_{2k+1}(a)| < \frac{a^{-3/4}}{4\pi}\int_1^\infty \{a^{1/4}\phi(t) + a^{-1/4}\psi(t)\}\,\frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}}\,dt$$

and therefore we obtain

Theorem 4. *The remainder term $\epsilon_{2k+1}(a)$ defined in (3.4) satisfies the upper bound*

$$|\epsilon_{2k+1}(a)| < \mathcal{B}_{2k+1}(a), \qquad \mathcal{B}_{2k+1}(a) := \frac{a^{-3/4}(2k+1)!}{4\sqrt{2\pi}}\{E_{2k+1}(a) + E_{2k+1}(1/a)\}, \qquad (4.5)$$

where

$$E_{2k+1}(a) := a^{1/4}\Psi(a)U(2k+2, \tfrac{1}{2}, 2\pi a).$$

and $\Psi(a)$ is given by (2.5). The leading behaviour of $\mathcal{B}_{2k+1}(a)$ for large k and finite a is given by the right-hand side of (4.4).

[1] It is clear that this bound will not be sharp in the neighbourhood of $a \simeq 1$.

5. Numerical results To demonstrate the smallness of the remainder terms $\epsilon_{2k}(a)$ and $\epsilon_{2k+1}(a)$ we define the quantities

$$\mathcal{J}_{2k}(a) := J_{2k}(a) - \frac{1}{4\pi a}\Big\{\Big(\frac{1+a^{1/2}}{2}\Big)\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+1)}{\Gamma(2k+\frac{3}{2})} - {}_2F_1(-2k,1;\tfrac{3}{2};2)\Big\}$$

and

$$\mathcal{J}_{2k+1}(a) := J_{2k+1}(a) + \frac{1}{4\pi a}\Big\{\Big(\frac{1-a^{1/2}}{2}\Big)\sqrt{\frac{\pi}{2}}\,\frac{\Gamma(2k+2)}{\Gamma(2k+\frac{5}{2})} + {}_2F_1(-2k-1,1;\tfrac{3}{2};2)\Big\}.$$

In Tables 1–3 we present numerical values of these quantities compared with their bounds $\mathcal{B}_{2k}(a)$ and $\mathcal{B}_{2k+1}(a)$ for a range of k and three values of the parameter $a = O(1)$. It is seen that this bound agrees very well with the computed values of $\mathcal{J}_{2k}(a)$ and $\mathcal{J}_{2k+1}(a)$. The estimates in (4.4) and (4.5) show that the remainder terms are *exponentially small* for large k when $a = O(1)$. Consequently, the terms $T_{2k}(a)$ and $T_{2k+1}(a)$ in (2.10) and (3.3) approximate $J_{2k}(a)$ and $J_{2k+1}(a)$, respectively, to exponential accuracy in the large-k limit.

Now

$$J_{2k}(0) = J_{2k+1}(0) = \int_0^\infty \frac{x}{e^{2\pi x}-1}\,dx = \frac{1}{24};$$

but it is easily seen that $T_{2k}(a)$ and $T_{2k+1}(a)$ in (2.10) and (3.3) are $O(a^{-1})$ as $a \to 0$ and $O(a^{-1/2})$ as $a \to \infty$. Routine calculations show that the bounds $\mathcal{B}_{2k}(a)$ and $\mathcal{B}_{2k+1}(a)$ also possess the same behaviour in these limits. Consequently, the approximtions $T_{2k}(a)$ and $T_{2k+1}(a)$ will not be good for small or large values of the parameter a, although it is worth pointing out that the range of validity in a will increase as k increases.

Table 1: Values of $\mathcal{J}_{2k}(a)$ and the bound $\mathcal{B}_{2k}(a)$ for $\epsilon_{2k}(a)$ in (4.2) as a function of k when $a = 1$.

k	$\mathcal{J}_{2k}(1)$	$\mathcal{B}_{2k}(1)$	k	$\mathcal{J}_{2k}(1)$	$\mathcal{B}_{2k}(1)$
1	1.250×10^{-5}	1.253×10^{-5}	10	3.905×10^{-12}	3.913×10^{-12}
2	8.571×10^{-7}	8.588×10^{-7}	20	3.186×10^{-16}	3.193×10^{-16}
3	9.818×10^{-8}	9.838×10^{-8}	30	2.305×10^{-19}	2.309×10^{-19}
5	2.883×10^{-9}	2.888×10^{-9}	50	2.433×10^{-24}	2.438×10^{-24}

The approximation in (1.4) when $\alpha = a\pi$ (with $\alpha\beta = \pi^2$) yields

$$J_{2k}(a) \simeq -\frac{F}{4\pi a}\left\{1 - \left(1 + a^2 + \frac{2\pi a}{3F}\right)^{1/4}\right\}, \qquad F := {}_2F_1(-2k,1;\tfrac{3}{2};2). \quad (5.1)$$

Table 2: Values of $\mathcal{J}_{2k}(a)$ and the bound $\mathcal{B}_{2k}(a)$ as a function of k when $a = 2$ and $a = 0.50$.

k	$\mathcal{J}_{2k}(2)$	$\mathcal{B}_{2k}(2)$	k	$\mathcal{J}_{2k}(2)$	$\mathcal{B}_{2k}(2)$
1	5.987×10^{-5}	6.364×10^{-5}	10	9.509×10^{-10}	1.011×10^{-9}
2	7.856×10^{-6}	8.355×10^{-6}	20	1.075×10^{-12}	1.143×10^{-12}
3	1.563×10^{-6}	1.662×10^{-6}	30	6.016×10^{-15}	6.398×10^{-15}
5	1.162×10^{-7}	1.236×10^{-7}	50	1.668×10^{-18}	1.774×10^{-18}
k	$\mathcal{J}_{2k}(\frac{1}{2})$	$\mathcal{B}_{2k}(\frac{1}{2})$	k	$\mathcal{J}_{2k}(\frac{1}{2})$	$\mathcal{B}_{2k}(\frac{1}{2})$
1	1.693×10^{-4}	1.800×10^{-4}	10	2.689×10^{-9}	2.860×10^{-9}
2	2.222×10^{-5}	2.363×10^{-5}	20	3.040×10^{-12}	3.234×10^{-12}
3	4.420×10^{-6}	4.700×10^{-6}	30	1.702×10^{-14}	1.810×10^{-14}
5	3.287×10^{-7}	3.496×10^{-7}	50	4.719×10^{-18}	5.019×10^{-18}

Table 3: Values of $\mathcal{J}_{2k+1}(a)$ and the bound $\mathcal{B}_{2k+1}(a)$ as a function of k when $a = 2$ and $a = 0.50$.

k	$\mathcal{J}_{2k+1}(2)$	$\mathcal{B}_{2k+1}(2)$	k	$\mathcal{J}_{2k+1}(2)$	$\mathcal{B}_{2k+1}(2)$
0	2.230×10^{-4}	2.376×10^{-4}	10	6.340×10^{-10}	6.743×10^{-10}
1	2.018×10^{-5}	2.147×10^{-5}	20	8.067×10^{-13}	8.579×10^{-13}
2	3.376×10^{-6}	3.591×10^{-6}	30	4.760×10^{-15}	5.063×10^{-15}
5	6.603×10^{-8}	7.022×10^{-8}	40	6.287×10^{-17}	6.686×10^{-17}
k	$\mathcal{J}_{2k+1}(\frac{1}{2})$	$\mathcal{B}_{2k+1}(\frac{1}{2})$	k	$\mathcal{J}_{2k+1}(\frac{1}{2})$	$\mathcal{B}_{2k+1}(\frac{1}{2})$
0	6.307×10^{-4}	6.720×10^{-4}	10	1.793×10^{-9}	1.907×10^{-9}
1	5.708×10^{-5}	6.072×10^{-5}	20	2.282×10^{-12}	2.427×10^{-12}
2	9.548×10^{-6}	1.106×10^{-5}	30	1.346×10^{-14}	1.432×10^{-14}
5	1.867×10^{-7}	1.986×10^{-7}	40	1.778×10^{-16}	1.891×10^{-16}

This yields the limiting behaviours

$$J_{2k}(a) \simeq \frac{1}{24} + a\left(\frac{F}{16\pi} - \frac{\pi}{96F}\right) + O(a^2) \qquad (a \to 0)$$

and

$$J_{2k}(a) \simeq \frac{F}{4\pi\sqrt{a}}\left\{1 - \frac{1}{\sqrt{a}} + \frac{\pi}{6aF} + O(a^{-3/2})\right\} \qquad (a \to \infty).$$

The approximation (5.1) is found to be quite accurate in the limits of small and large a, with k finite. However, the accuracy is not good when $a = O(1)$.

For example, when $a = 1$ and $k = 5, 10$ the approximation (5.1) yields absolute relative errors of 8.8% and 19.2%, respectively; and this error increases as k increases, in marked contrast to the approximations in (2.10) and (3.3).

As a final remark, it is doubtful that the remainder terms $\epsilon_{2k}(a)$ and $\epsilon_{2k+1}(a)$ can be expressed in simple closed forms.

References

[1] B.C. Berndt and R.J. Evans, Some elegant approximations and asymptotic formulas of ramanujan, J. Comp. Appl. Math. **37** (1991), 35–41.

[2] A. Dixit, A. Roy and A. Zaharescu, Error functions, Mordell integrals and an integral analogue of a partial theta function, Acta Arith. **177** (2017) 1–37.

[3] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974; Reprinted in A.K. Peters, Massachussets, 1997.

[4] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[5] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927.

[6] S. Ramanujan, *Notebooks*, Tata Inst. Fund. Res., Bombay, 1957 (2 volumes).

[7] E.T. Whittaker and G.N. Watson, *Modern Analysis*, Cambridge University Press, Cambridge, 1952.

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