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A generalized statistical convergence via ideals in 2-normed spaces

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Abstract

In this paper we introduce and investigate lacunary convergence, $\mathcal{I}$-statistical convergence, strongly lacunary convergence, strongly $\mathcal{I}$-lacunary convergence, strongly $\mathcal{I}^*$-lacunary convergence and strongly $\mathcal{I}$-lacunary Cauchy sequences in 2-normed spaces and study their certain properties. Also, we give the notion of $\mathcal{I}$-statistically pre-Cauchy sequences in 2 normed. We mainly show that $\mathcal{I}$-statistical convergence implies $\mathcal{I}$-statistically pre-Cauchy condition and give certain sufficient conditions for the converse to be true.

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1 Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [8] and Schoenberg [26].

The notion of $\mathcal{I}$-convergence was studied at initial stage by Kostyrko et al. [2]. Kostyrko et al. [3] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Later on it was studied by Šalát et al. [4], Tripathy and Hazarika [5] and many others. Recently, Das et al. [1] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal.

Fridy and Orhan [7] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [9]. Tripathy et al. [13] introduced the concepts of $\mathcal{I}$-lacunary convergent sequences. Bakery and Mohammed [15] introduced lacunary mean ideal convergence in generalized random n-normed spaces. Also, Yamancı and Gürdal [27] introduced the notion lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy in the topology induced by random n-normed spaces and prove some important results.

The concept of 2-normed spaces was initially introduced by Gähler ([14], [17]) in the 1960's and and has been developed extensively in different subjects by others ([22], [23], [30]).

Gürdal and Pehlivan [16] studied statistical convergence in 2-normed spaces. They showed that some properties of statistical convergence of real number sequences also hold for sequences in 2-normed spaces. They also defined the notion of a statistical Cauchy sequence in 2-normed spaces, obtained a criteria for a sequence in a 2-normed spaces to be a statistical Cauchy sequence. Şahiner et al. [24] and Gürdal [21] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [20] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces.

Das and Savaş [29] continued their investigation of $\mathcal{I}$-statistical convergence and introduce the notion of $\mathcal{I}$-statistically pre-Cauchy sequences in line of [28].

We first recall some basic definitions and notions which will be needed.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,

(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,

(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$,

(ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,

(iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$, and by a sequence we always mean a sequence of real numbers.

An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of $\mathcal{I}$, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let X be a real vector space of dimension greater than 1, and $\|.,.\|$ be a non-negative real-valued function on $X \times X$ satisfiying the following conditios:

$G1)$ $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors,
$G2)$ $\|x, y\| = \|y, x\|$ for all x, y in X,
$G3)$ $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ where α is real,
$G4)$ $\|x + y, z\| \leq \|x, y\| + \|y, z\|$ for all x, y, z in X.

$\|., .\|$ is called a 2-norm on X and the pair $(X, \|., .\|)$ is called a linear 2-normed space.

Given a 2-normed space $(X, \|., .\|)$, one can derive topology for it via the following definition of the limit of a seqeunce: a sequence $(x_k)_{k\in\mathbb{N}}$ in X is said to be convergent to l in X if $\lim_{k\to\infty} \|x_k - x, z\| = 0$ This can be written by the formula:

$$(\forall z \in X) \, (\forall \varepsilon > 0) \, (\exists n_0 \in \mathbb{N}) \, (\forall n \geq n_0) \qquad \|x_k - l, z\| < \varepsilon$$

We write it as

$$x_k \overset{\|.,.\|_X}{\to} l.$$

A sequence $(x_k)_{n\in\mathbb{N}}$ in X is said to be statistical convergent to l in X if for each $\varepsilon > 0$ and for each nonzero $z \in X$, the set $\{k \in \mathbb{N} : \|x_k - l, z\| \geq \varepsilon\}$ has natural density zero in other words

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : \|x_k - l, z\| > \varepsilon\}\right| = 0.$$

This implies that $st\text{-}\lim_{k\to\infty} \|x_k, z\| = \|l, z\|$.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_k)_{k\in\mathbb{N}}$ of X is said to be $\mathcal{I}$-convergent to l, in X if for each $\varepsilon > 0$ and $z \in X$ the set $\{k \in \mathbb{N} : \|x_k - l, z\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

If (x_k) is $\mathcal{I}$-convergent to l, then we write $\mathcal{I}\text{-}\lim_{n\to\infty} \|x_k - l, z\| = 0$ or $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k, z\| = \|l, z\|$. The number l is $\mathcal{I}$-limit of the sequence (x_k).

2 Main Results

We now introduce the main definitions of this paper.

Definition 2.1 *A sequence $x = (x_k)$ in X is said to be lacunary statistical convergent to $l \in X$ if for each $\varepsilon > 0$ and for each nonzero $z \in X$*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \|x_k - l, z\| > \varepsilon\}\right| = 0.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\to} l \, (S_\theta) \, .$$

Definition 2.2 *A sequence $x = (x_k)$ in X is said to be $\mathcal{I}$-statistical convergent to $l \in X$ if for each $\varepsilon > 0$, $\delta > 0$ and for each nonzero $z \in X$*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \| x_k - l, z \| > \varepsilon \right\} \right| > \delta \right\} \in \mathcal{I}.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\rightarrow} l \left(S \left(\mathcal{I} \right) \right).$$

Definition 2.3 *A sequence $x = (x_k)$ in X is said to be $\mathcal{I}$-lacunary statistical convergent to $l \in X$ if for each $\varepsilon > 0$, $\delta > 0$ and for each nonzero $z \in X$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \| x_k - l, z \| > \varepsilon \right\} \right| > \delta \right\} \in \mathcal{I}.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\rightarrow} l \left(S_\theta \left(\mathcal{I} \right) \right).$$

Definition 2.4 *A sequence $x = (x_k)$ in X is said to be strongly lacunary convergent to $l \in X$ if for each nonzero $z \in X$*

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \| x_k - l, z \| = 0.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\rightarrow} l \left[N_\theta \right].$$

Definition 2.5 *A sequence $x = (x_k)$ in X is said to be strongly $\mathcal{I}$-lacunary convergent to $l \in X$ if for each $\varepsilon > 0$ and for each nonzero $z \in X$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \| x_k - l, z \| \geq \varepsilon \right\} \in \mathcal{I}.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\rightarrow} l \left(N_\theta \left[\mathcal{I} \right] \right).$$

Definition 2.6 *A sequence $x = (x_k)$ in X is said to be $\mathcal{I}^*$-lacunary convergent to $l \in X$ if and only if there exists a set $M = \{ m_i : m_i < m_{i+1} \ , \ i \in \mathbb{N} \} \subset \mathbb{N}$ such that $M' = \{ r \in \mathbb{N} : m_k \in I_r \} \in \mathcal{F} \left(\mathcal{I} \right)$ for each nonzero $z \in X$*

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \| x_{m_k}, z \| = \| l, z \|.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\rightarrow} l \left(N_\theta \left(\mathcal{I}^* \right) \right).$$

Definition 2.7 *A sequence $x = (x_k)$ in X is said to be strongly $\mathcal{I}^*$-lacunary convergent to $l \in X$ if and only if there exists a set $M = \{m_i : m_i < m_{i+1}\ , \ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ for each nonzero $z \in X$*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \|x_{m_k} - l, z\| = 0.$$

We write it as

$$x_k \overset{\|.,.\|_X}{\to} l\left(N_\theta\left[\mathcal{I}^*\right]\right).$$

Theorem 2.1 *If (x_k) is strongly lacunary convergent $l \in X$, then it is strongly $\mathcal{I}$-lacunary convergent to l.*

Proof 2.1 *Let (x_k) is strongly lacunary convergent $l \in X$. For each $\varepsilon > 0$ and $z \in X$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that*

$$\frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| < \varepsilon,$$

for all $k \geq k_0$. Then

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| \geq \varepsilon \right\} \subset \{1, 2, ..., k_0 - 1\}.$$

Since $\mathcal{I}$ is admissible ideal we have $\{1, 2, ..., k_0 - 1\} \in \mathcal{I}$ and so $A(\varepsilon) \in \mathcal{I}$. This completes the proof.

Theorem 2.2 *If (x_k) is strongly $\mathcal{I}^*$-lacunary convergent to $l \in X$, then it is strongly $\mathcal{I}$-lacunary convergent to l.*

Proof 2.2 *Suppose that (x_k) is strongly $\mathcal{I}^*$-lacunary convergent to $l \in X$. Then there exists a set $M = \{m_i : m_i < m_{i+1}\ , \ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$*

$$\frac{1}{h_r} \sum_{k \in I_r} \|x_{m_k} - l, z\| < \varepsilon,$$

for each $\varepsilon > 0$ and for all $k > k_0 = k_0(\varepsilon) \in \mathbb{N}$. Hence, for each $\varepsilon > 0$ and $z \in X$ we have

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \|x_{m_k} - l, z\| \geq \varepsilon \right\} \subset H \cup \{1, 2, ..., k_0 - 1\},$$

for $\mathbb{N} \backslash M' = H \in \mathcal{I}$. Since $\mathcal{I}$ is an admissible ideal we have

$$H \cup \{m_1, m_2, ..., m_{k_0 - 1}\} \in \mathcal{I}$$

and so $A(\varepsilon) \in \mathcal{I}$. Hence, this completes the proof.

Theorem 2.3 *Let $\mathcal{I}$ be an admissible ideal with property (AP). (x_k) is strongly $\mathcal{I}$-lacunary convergent to l implies (x_k) is strongly $\mathcal{I}^*$-lacunary convergent to l.*

Proof 2.3 *Suppose that (x_k) is strongly $\mathcal{I}$-lacunary convergent to $l \in X$. Then, for each $\varepsilon > 0$ and $z \in X$*

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| \geq \varepsilon \right\} \in \mathcal{I}.$$

Put

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| \geq 1 \right\},$$

$$A_p = \left\{ r \in \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| < \frac{1}{p-1} \right\}$$

for $p \geq 2$ and $p \in \mathbb{N}$. Obviously $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By property (AP) there is a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is a finite set for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. We prove that for each $z \in X$

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \|x_{m_k} - l, z\| = 0,$$

for $M = \mathbb{N} \setminus B$, $M = \{m = (m_i) : m_i < m_{i+1}\ ,\ i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$.

Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then for each $z \in X$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} A_j.$$

Since $A_j \Delta B_j$, $j = 1, 2, ...$is a finite set for $j \in \{1, 2, ..., q-1\}$, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} A_j \right) \cap \{k \in \mathbb{N} : k \geq k_0\} = \left(\bigcup_{j=1}^{q-1} B_j \right) \cap \{k \in \mathbb{N} : k \geq k_0\}$$

If $k \geq k_0$ and $k \notin B$, then $k \notin \bigcup_{j=1}^{q-1} B_j$ and so $k \notin \bigcup_{j=1}^{q-1} A_j$. Thus, for each $z \in X$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} \|x_k - l, z\| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|x_{m_k} - l, z\| = 0.$$

Hence, we have (x_k) *is strongly* $\mathcal{I}^*$*-lacunary convergent to* l*. This completes the proof.*

Definition 2.8 *We say that the sequence* (x_k) *is strongly* $\mathcal{I}$*-lacunary Cauchy sequence in* X*, if for each* $\varepsilon > 0$ *for* $z \in X$*, there exists a number* $N = N(\varepsilon, z) \in \mathbb{N}$ *such that*

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \left\| x_k - x_{N(\varepsilon,z)}, z \right\| \geq \varepsilon \right\} \in \mathcal{I}.$$

Theorem 2.4 *If* (x_k) *is strongly* $\mathcal{I}$*-lacunary convergent in* X*, then* (x_k) *is strongly* $\mathcal{I}$*-lacunary Cauchy sequence in* X*.*

Proof 2.4 *Let* (x_k) *is strongly* $\mathcal{I}$*-lacunary convergent to* l*. Then for each* $\varepsilon > 0$ *and* $z \in X$*, we have*

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \|x_k - l, z\| \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ *is admissible ideal, the set*

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \|x_k - l, z\| < \frac{\varepsilon}{2} \right\}$$

is non-empty and belongs to $\mathcal{F}(\mathcal{I})$*. So, we can choose positive integer* r *such that* $r \notin A\left(\frac{\varepsilon}{2}, z\right)$*, we have*

$$\frac{1}{h_r} \sum_{k_0\in I_r} \|x_{k_0} - l, z\| < \frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon, z) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k,k_0\in I_r} \|x_k - x_{k_0}, z\| \geq \varepsilon \right\}.$$

We show that $B(\varepsilon, z) \subset A\left(\frac{\varepsilon}{2}, z\right)$*. Let* $r \in B(\varepsilon, z)$ *then we have*

$$\varepsilon \leq \tfrac{1}{h_r} \sum_{k,k_0\in I_r} \|x_k - x_{k_0}, z\| \leq \tfrac{1}{h_r} \sum_{k\in I_r} \|x_k - l, z\| + \tfrac{1}{h_r} \sum_{k_0\in I_r} \|x_{k_0} - l, z\|$$

$$< \tfrac{1}{h_r} \sum_{k\in I_r} \|x_k - l, z\| + \tfrac{\varepsilon}{2}.$$

This implies that

$$\frac{1}{h_r}\sum_{k\in I_r}\|x_k-l,z\|>\frac{\varepsilon}{2}$$

and therefore $r\in A\left(\frac{\varepsilon}{2},z\right)$. *Hence, we have* $A\left(\frac{\varepsilon}{2},z\right)\subset B(\varepsilon,z)$. *This shows that* (x_k) *is strongly* $\mathcal{I}$*-lacunary Cauchy sequence.*

Now, we give the notion of $\mathcal{I}$-statistically pre-Cauchy in 2 normed spaces. We mainly show that $\mathcal{I}$-statistical convergence implies $\mathcal{I}$-statistically pre-Cauchy condition and give certain sufficient conditions for the converse to be true.

Definition 2.9 *A sequence* (x_k) *is said to be* $\mathcal{I}$*-statistical pre-Cauchy sequence in 2-normed spaces, if for any* $\varepsilon>0$, $\delta>0$ *and for* $z\in X$,

$$\left\{n\in\mathbb{N}:\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k\le n:\|x_k-x_j,z\|>\varepsilon\right\}\right|>\delta\right\}\in\mathcal{I}.$$

Theorem 2.5 *An* $\mathcal{I}$*-statistical convergent sequence in 2-normed spaces is* $\mathcal{I}$*-statistical pre-Cauchy.*

Proof 2.5 *Let* (x_k) *be* $\mathcal{I}$*-statistical convergent to* l. *Let* $\varepsilon,\delta>0$ *be given.Now*

$$T=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\|x_k-l,z\|\ge\frac{\varepsilon}{2}\right\}\right|\ge\delta\right\}\in\mathcal{I}.$$

Then for $n\in T^c$ *where* c *stands for the complement,*

$$\frac{1}{n}\left|\left\{k\le n:\|x_k-l,z\|\ge\frac{\varepsilon}{2}\right\}\right|<\delta$$

$$\text{i.e. }\frac{1}{n}\left|\left\{k\le n:\|x_k-l,z\|<\frac{\varepsilon}{2}\right\}\right|>1-\delta.$$

Writing $B_n=\left\{k\le n:\|x_k-l,z\|<\frac{\varepsilon}{2}\right\}$ *we observe that for* $j,k\in B_n$

$$\|x_k-x_j,z\|\le\|x_k-l,z\|+\|x_j-l,z\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence

$$B_n\times B_n\subset\left\{(j,k)\,,\ j,k\le n:\|x_k-x_j,z\|<\varepsilon\right\}$$

which implies

$$\left[\frac{|B_n|}{n}\right]^2\le\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k\le n:\|x_k-x_j,z\|<\varepsilon\right\}\right|.$$

Thus for all $n\in T^c$,

$$\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k\le n:\|x_k-x_j,z\|<\varepsilon\right\}\right|\ge\left[\frac{|B_n|}{n}\right]^2>(1-\delta)^2$$

$$i.e.\ \frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right| < 1 - (1-\delta)^2.$$

Let $\delta_1 > 0$ *be given. Choosing* $\delta > 0$ *so that* $1 - (1-\delta)^2 < \delta_1$ *we see that* $\forall n \in T^c$

$$\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right| < \delta_1$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right| \ge \delta_1\right\} \subset T.$$

Since $T \in \mathcal{I}$, *so*

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right| \ge \delta_1\right\} \in \mathcal{I}$$

and this completes the proof the theorem.

In the following we give a necessary and sufficient condition for a sequence to be $\mathcal{I}$-statistical pre-Cauchy in 2-normed spaces.

Theorem 2.6 *Let* (x_k) *be a bounded sequencey in 2-normed spaces. Then* x *is* $\mathcal{I}$*-statistical pre-Cauchy in* X *if and only if*

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k\le n} \|x_k - x_j, z\| = 0.$$

Proof 2.6 *First suppose that*

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k\le n} \|x_k - x_j, z\| = 0.$$

Note that for any $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *we have*

$$\frac{1}{n^2} \sum_{j,k\le n} \|x_k - x_j, z\| \ge \varepsilon.\left(\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right|\right).$$

Hence for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \le n : \|x_k - x_j, z\| \ge \varepsilon\right\}\right| \ge \delta\right\}$$

$$\subset \left\{n \in \mathbb{N} : \tfrac{1}{n^2} \sum_{j,k\le n} \|x_k - x_j, z\| \ge \varepsilon\delta\right\}.$$

Since $\mathcal{I} - \lim_{n\to\infty} \frac{1}{n^2} \sum\limits_{j,k\leq n} \|x_k - x_j, z\| = 0$ *so the set on the right hand belongs to* $\mathcal{I}$ *which implies that*

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \left|\left\{(j,k)\,,\ j,k \leq n : \|x_k - x_j, z\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

This proves that x *is* $\mathcal{I}$*-statistical pre-Cauchy in* X*.*

Conversely suppose that x *is* $\mathcal{I}$*-statistical pre-Cauchy in* X*. Since* x *is bounded,* $\exists$ *a* $B > 0$ *such that* $|x_k| \leq B$ *for* $\forall k \in \mathbb{N}$*. Let* $\delta > 0$ *be given. For each* $n \in \mathbb{N}$*,*

$$\frac{1}{n^2} \sum_{j,k\leq n} \|x_k - x_j, z\| \leq \frac{\varepsilon}{2} + 2B\left(\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \leq n : \|x_k - x_j, z\| \geq \frac{\varepsilon}{2}\right\}\right|\right).$$

Since x *is* $\mathcal{I}$*-statistical pre-Cauchy in* X*, for* $\delta > 0$

$$K = \left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \leq n : \|x_k - x_j, z\| \geq \frac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Then for $n \in K^c$

$$\frac{1}{n^2}\left|\left\{(j,k)\,,\ j,k \leq n : \|x_k - x_j, z\| \geq \frac{\varepsilon}{2}\right\}\right| < \delta$$

and so

$$\frac{1}{n^2} \sum_{j,k\leq n} \|x_k - x_j, z\| \leq \frac{\varepsilon}{2} + 2B\delta.$$

Let $\delta_1 > 0$ *be given. Then choosing* ε*,* $\delta > 0$ *so that* $\frac{\varepsilon}{2} + 2B\delta < \delta_1$ *we see that* $\forall n \in K^c$*,*

$$\frac{1}{n^2} \sum_{j,k\leq n} \|x_k - x_j, z\| < \delta_1$$

i.e.

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \sum_{j,k\leq n} \|x_k - x_j, z\| \geq \delta_1\right\} \subset K \in \mathcal{I}.$$

This proves the necessity of the condition.

References

[1] Das P., Savaş, E. and Ghosal, S. Kr., On generalized of certain summability methods using ideals, *Appl. Math. Letter*, 36 (2011) 1509-1514.

[2] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-Convergence, Real Anal. Exchange **26**(2) (2000) 669–686.

[3] P. Kostyrko, M. Macaj, T. Šalát, M. Sleziak, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math. Slovaca 55 (2005) 443–464.

[4] T. Šalát, B.C. Tripathy, M. Ziman, On some properties of $\mathcal{I}$-convergence,Tatra Mt. Math. Publ. 28 (2004) 279–286.

[5] B.C. Tripathy, B. Hazarika, $\mathcal{I}$-monotonic and $\mathcal{I}$-convergent sequences, Kyungpook Math. J. 51 (2011) 233–239, http://dx.doi.org/10.5666/KMJ.2011.51.2.233.

[6] Das P., Savaş E., Ghosal S.K., On generalizations of certain summability methods using ideals, Appl. Math. Lett. 24 (2011), 1509–1514.

[7] J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1) (1993) 43–51.

[8] Fast H., Sur la convergence statistique, Colloq. Math. **2** (1951) 241–244.

[9] J.A. Fridy, C. Orhan, Lacunary statistical summability, J. Math. Anal. Appl. 173 (1993) 497–504, MR 95f :40004.

[10] Hazarika B., Lacunary ideal convergence of multiple sequences, Journal of the Egyptian Mathematical Society, http://dx.doi.org/10.1016/j.joems.(2014), 07.002.

[11] A.A. Bakery, Operator ideal of Cesáro type sequence spaces involving Lacunary sequence, Abst. Appl. Anal., vol. 2014, 2014, 6 pp. Article ID 419560, http://dx.doi.org/10.1155/2014/ 419560.

[12] B.C. Tripathy, B. Hazarika, B. Choudhary, Lacunary $\mathcal{I}$-convergentsequences, in: Real Analysis Exchange Summer Symposium, 2009, pp. 56–57.

[13] B.C. Tripathy, B. Hazarika, B. Choudhary, Lacunary $\mathcal{I}$-convergent sequences, Kyungpook Math. J. 52 (4) (2012) 473– 482.

[14] S. Gähler, 2-metrische Räume und ihre topologische Struktur, Math. Nachr., 26(1963), 115-148.

[15] A.A. Bakery, M.M. Mohammed, On lacunary mean ideal convergence in generalized random n-normed spaces, Abstract Appl. Anal. 2014 (2014) 11, http://dx.doi.org/10.1155/2014/101782. Article ID: 101782.

[16] M. Gürdal and S. Pehlivan, The Statistical Convergence in 2-Banach Spaces, Thai J. Math., 2(1) (2004) 107-113.

[17] S. Gähler, 2-normed spaces, Math. Nachr. 28 (1964), 1–43.

[18] H. Gunawan, M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (10) (2001) 631–639.

[19] H. Gunawan, M. Mashadi, On finite dimensional 2-normed spaces, Soochow J. Math. 27 (3) (2001) 321–329.

[20] M. Gürdal, I. Açık, On $\mathcal{I}$-Cauchy sequences in 2-normed spaces, Math. Inequal. Appl. 11 (2) (2008) 349–354.

[21] M. Gürdal, On ideal convergent sequences in 2-normed spaces, Thai J. Math. 4 (1) (2006) 85–91.

[22] S. Sarabadan, F.A. Arani, S. Khalehoghli, A condition for the equivalence of $\mathcal{I}$ and $\mathcal{I}^*$-convergence in 2-normed spaces, Int. J. Comtemp. Math. Sci. 6 (43) (2011) 2147–2159.

[23] Sarabadan, S. Talebi, Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces, Internat. J. Math. Math. Sci. 2011 (2011) 10 pages.

[24] PA. Sahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math. 11 (2007) 1477–1484.

[25] B.C. Tripathy, M. Sen, S. Nath, $\mathcal{I}$-convergence in probabilistic n-normed space, Soft Comput. 16 (2012) 1021–1027.

[26] Schoenberg I. J., The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66 (1959) 361-375.

[27] U. Yamancı, and M. Gürdal, Journal of Mathematics, 2013, 1-8 (2013), Article ID 868457.

[28] J. Connor, J. A. Fridy and J. Kline, Statistically pre-Cauchy sequences, Analysis, 14 (1994) 311-317.

[29] P. Das, E. Savaş, On $\mathcal{I}$-Statistically pre-Cauchy sequences, Thai J. Math. Vol. 18, No. 1, pp. 115-126, February 2014 DOI: 10.11650/tjm.18.2014.3157

[30] W. Raymond, Y. Freese , J. Cho, Geometry of linear 2-normed spaces, N.Y. Nova Science Publishers, Huntington, 2001.

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