3-Lie bialgebras \((L_c, C_d)\) and \((L_c, C_e)\)

BAI Ruipu

College of Mathematics and Information Science, Hebei University, Baoding, 071002, China
e-mail: bairuipu@hbu.edu.cn

GUO Weiwei

College of Mathematics and Information Science, Hebei University, Baoding, 071002, China

Abstract

In this paper, we discuss the structure of four dimensional 3-Lie bialgebras of type \((L_{c_i}, C_d)\) and \((L_{c_i}, C_e)\) for \(i = 1, 2, 3\). It is proved that there do not exist 3-Lie bialgebras of types \((L_{c_i}, C_d)\) and \((L_{c_i}, C_e)\) for \(i = 1, 2\) (Theorem 3.2), and there exist only three classes of 3-Lie bialgebras of types \((L_{c_3}, C_d)\) (Theorem 3.3), and two classes of 3-Lie bialgebras of types \((L_{c_3}, C_e)\) (Theorem 3.4).

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1 Preliminaries

A 3-Lie algebra [1] is a vector space \(L\) endowed with a linear multiplication \(\mu : L^3 \to L\) satisfying that, for all \(x, y, z, u, v \in L\),

\[
\mu(u, v, \mu(x, y, z)) = \mu(x, y, \mu(u, v, z)) + \mu(y, z, \mu(u, v, x)) + \mu(z, x, \mu(u, v, y)).
\]

For defining 3-Lie coalgebras, we need to define following linear maps \(\omega_i : L \otimes L \otimes L \otimes L \to L \otimes L \otimes L \otimes L, 1 \leq i \leq 3\), by

\[
\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5,
\]

\[
\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3,
\]

\[
\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4.
\]

A 3-Lie coalgebra \((L, \Delta)\) [2] is a vector space \(L\) with a linear map \(\Delta : L \to L \otimes L \otimes L\) satisfying

\[
Im(\Delta) \subset L \wedge L \wedge L, \text{ and } (1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \Delta)\Delta = 0.
\]

Let \((L_1, \Delta_1)\) and \((L_2, \Delta_2)\) be 3-Lie coalgebras. If there is a linear isomorphism \(\varphi : L_1 \to L_2\) satisfying \((\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e)), \text{ for all } e \in L_1,
then \((L_1, \Delta_1)\) is isomorphic to \((L_2, \Delta_2)\), and \(\varphi\) is called a 3-Lie coalgebra isomorphism, where \((\varphi \otimes \varphi \otimes \varphi) \sum_i (a_i \otimes b_i \otimes c_i) = \sum_i \varphi(a_i) \otimes \varphi(b_i) \otimes \varphi(c_i)\).

A 3-Lie bialgebra[2] is a triple \((L, \mu, \Delta)\) such that

1. \((L, \mu)\) is a 3-Lie algebra with the multiplication \(\mu : L \wedge L \wedge L \to L\),
2. \((L, \Delta)\) is a 3-Lie coalgebra with \(\Delta : L \to L \wedge L \wedge L\),
3. \(\Delta\) and \(\mu\) satisfy the following identity, for \(x, y, u, v, w \in L\),

\[
\Delta\mu(x, y, z) = \text{ad}_\mu^{(3)}(x, y)\Delta(z) + \text{ad}_\mu^{(3)}(y, z)\Delta(x) + \text{ad}_\mu^{(3)}(z, x)\Delta(y),
\]

where \(\text{ad}_\mu^{(3)}(x, y), \text{ad}_\mu^{(3)}(z, x), \text{ad}_\mu^{(3)}(y, z) : L \otimes L \otimes L \to L \otimes L \otimes L\) are linear maps defined by (similar for \(\text{ad}_\mu^{(3)}(z, x)\) and \(\text{ad}_\mu^{(3)}(y, z)\))

\[
\text{ad}_\mu^{(3)}(x, y)(u \otimes v \otimes w) = (\text{ad}_\mu(x, y) \otimes 1 \otimes 1)(u \otimes v \otimes w) + (1 \otimes \text{ad}_\mu(x, y))(u \otimes v \otimes w)
\]

\[
= \mu(x, y) \otimes v \otimes w + u \otimes \mu(x, y, v) \otimes w + u \otimes v \otimes \mu(x, y, w).
\]

**Lemma 2.1** Let \((L, \mu)\) be a 4-dimensional 3-Lie algebra with \(\dim L \neq 0, 2, 4\), and \(e_1, e_2, e_3, e_4\) be a basis of \(L\). Then \(L\) is isomorphic to one and only one of the following

\[
L_{b_1} = \mu(e_2, e_3, e_4) = 1, \quad L_{b_2} = \mu(e_1, e_2, e_3) = 1,
\]

\[
L_d = \mu_d(e_2, e_3, e_4) = 1, \quad L_{e_1} = \mu_c(e_1, e_3, e_4) = 1, \quad L_{e_2} = \mu_c(e_1, e_2, e_4) = 1,
\]

\[
L_{e_3} = \mu_c(e_1, e_2, e_3) = 1.
\]

**2 3-Lie bialgebras of types \((L_c, C_d)\) and \((L_c, C_e)\)**

First we give the classification of 3-Lie coalgebras of the types \((L_c, C_b)\) and \((L_c, C_e)\).

**Lemma 3.1** [2][4] Let \((L, \Delta)\) be a 4-dimensional 3-Lie coalgebra with \(m\)-dimensional derived algebra \((m \geq 3)\), and \(e_1, e_2, e_3, e_4\) be a basis of \(L\). Then \(L\) isomorphic to one and only one of the, \(C_d = (L, \Delta_d)\) and \(C_e = (L, \Delta_e)\),

\[
C_d. \quad \Delta_d(e^1) = e^2 \wedge e^3 \wedge e^4, \quad \Delta_d(e^2) = e^1 \wedge e^3 \wedge e^4, \quad \Delta_d(e^3) = e^1 \wedge e^2 \wedge e^4,
\]

\[
C_e. \quad \Delta_e(e^1) = e^2 \wedge e^3 \wedge e^4, \quad \Delta_e(e^2) = e^1 \wedge e^3 \wedge e^4, \quad \Delta_e(e^3) = e^1 \wedge e^2 \wedge e^4,
\]

\[
\Delta_e(e^4) = e^1 \wedge e^2 \wedge e^3.
\]

For convenience, in the following, for a 3-Lie bialgebra \((L, \mu, \Delta)\), if the 3-Lie algebra \((L, \mu)\) is the case \((L, \mu_c)\) in Lemma 2.1 and the 3-Lie coalgebra \((L, \Delta)\) is the case \((L, \Delta_d)\) and \((L, \Delta_e)\) in Lemma 3.1, then the 3-Lie bialgebra \((L, \mu_c, \Delta_d)\) and \((L, \mu_c, \Delta_e)\) are simply denoted by \((L_{c_1}, C_d)\) and \((L_{c_1}, C_e)\), which are called the 3-Lie bialgebras of type \((L_{c_1}, C_d)\) and \((L_{c_1}, C_e)\), respectively.

For a given 3-Lie algebra \(L\), in order to find all the 3-Lie bialgebra structures on \(L\), we should find all the 3-Lie coalgebra structures on \(L\) which are compatible with the 3-Lie algebra \(L\). Although a permutation of a basis of \(L\) gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebra.

**Theorem 3.2** There do not exist 3-Lie bialgebras of types \((L_{c_1}, C_d)\), \((L_{c_1}, C_e)\), \((L_{c_2}, C_d)\) and \((L_{c_2}, C_e)\).

**Proof** By Lemma 3.1 and [5], and a direct computation, we obtain that the following six isomorphic 3-Lie coalgebras of the type \(C_e\):

1. \(\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3\);
and twenty-four isomorphic 3-Lie coalgebras of the type $C_d$:
(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;
(2) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;
(3) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_3 \wedge e_1 \wedge e_4$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;
(4) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;
(5) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_4 \wedge e_1 \wedge e_3$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;
(6) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1$, $\Delta(e_3) = e_2 \wedge e_1 \wedge e_3$;

which are incompatible with the 3-Lie algebra $L_{c_j}, j = 1, 2$. Therefore, there do not exist 3-Lie bialgebras of types $(L_{c_i}, C_d)$, $(L_{c_i}, C_e)$, for $i = 1, 2$. The proof is complete.

Theorem 3.3 The non-equivalent 3-Lie bialgebras of the type $(L_{c_3}, C_d)$ are as follows:

$(L_{c_3}, C_d, \Delta_1) : \Delta_1(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta_1(e_2) = e_1 \wedge e_3 \wedge e_4$, $\Delta_1(e_3) = e_1 \wedge e_2 \wedge e_4$;
$(L_{c_3}, C_d, \Delta_2) : \Delta_2(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta_2(e_2) = e_3 \wedge e_1 \wedge e_4$, $\Delta_2(e_3) = e_2 \wedge e_1 \wedge e_4$;
$(L_{c_3}, C_d, \Delta_3) : \Delta_3(e_1) = e_2 \wedge e_3 \wedge e_4$, $\Delta_3(e_3) = e_1 \wedge e_2 \wedge e_4$, $\Delta_3(e_4) = e_1 \wedge e_2 \wedge e_3$. 

Proof From Theorem 3.2, all the twenty-four cases of 3-Lie coalgebras of the type $C_d$ are compatible with the 3-Lie algebra $L_{c_3}$. And we have 3-Lie bialgebras isomorphisms

$$(L_{c_3}, C_d) :$$

$(1) \rightarrow (3), (2) \rightarrow (6), (4) \rightarrow (5), (7) \rightarrow (9), (8) \rightarrow (11), (10) \rightarrow (12),$

$(13) \rightarrow (24), (14) \rightarrow (23), (15) \rightarrow (22), (16) \rightarrow (21), (17) \rightarrow (19) :$

$f(e_1) = e_2, f(e_2) = e_1, f(e_3) = e_3, f(e_4) = e_4;$

$$(L_{c_3}, C_d) :$$

$(18), (15) \rightarrow (16) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = e_3, f(e_4) = e_4;$(1)

$(1) \rightarrow (10), (2) \rightarrow (11), (4) \rightarrow (7), (13) \rightarrow (14) :$

$f(e_1) = e_1, f(e_2) = e_2, f(e_3) = e_4, f(e_4) = -e_3;$

$(1) \rightarrow (4) : f(e_1) = e_2, f(e_2) = -e_1, f(e_3) = e_3, f(e_4) = e_4;$

$(19) \rightarrow (20) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_1, f(e_2) = -\frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_2,$

$f(e_3) = \frac{\sqrt{2}}{2}(1 - \sqrt{-1})e_3, f(e_4) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_4;$

$(13) \rightarrow (15) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_1, f(e_2) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_2,$

$f(e_3) = -\frac{\sqrt{2}}{2}(1 - \sqrt{-1})e_3, f(e_4) = -\frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_4;$

$(13) \rightarrow (20) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_1, f(e_2) = -\frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_2,$

$f(e_3) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_3, f(e_4) = \frac{\sqrt{2}}{2}(\sqrt{-1} - 1)e_3.$

Since for any 3-Lie algebra isomorphism

$$h : L_{c_3} \rightarrow L_{c_3} : h(e_j) = \sum_{k=1}^{4} a_{jk} e_k, a_{jk} \in F, j = 1, 2, 3, 4, a_{jk} \in F,$$

we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, and $det \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = 1$. By a direct computation $h$ is a non-isomorphism of the 3-Lie coalgebra (1) onto 3-Lie coalgebra (2). It follows that $(L_{c_3}, C_d, \Delta_1)$ and $(L_{c_3}, C_d, \Delta_2)$ are non-equivalent.

Summarizing above discussions, we obtain that the non-equivalent 3-Lie bialgebras are $(L_{c_3}, C_d, \Delta_1)$, $(L_{c_3}, C_d, \Delta_2)$ and $(L_{c_3}, C_d, \Delta_3)$. The proof is complete.

Theorem 3.4 The non-equivalent 3-Lie bialgebras of the type $(L_{c_3}, C_e)$ are as follows:

$$(L_{c_3}, C_e, \Delta_1) : \Delta_1(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_1(e_2) = e_1 \wedge e_3 \wedge e_4,$$

$\Delta_1(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta_1(e_4) = e_1 \wedge e_2 \wedge e_3;$

$$(L_{c_3}, C_e, \Delta_2) : \Delta_2(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_2(e_2) = e_1 \wedge e_4 \wedge e_3,$$

$\Delta_2(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta_2(e_4) = e_1 \wedge e_2 \wedge e_3.$

Proof By a direct computation, all the six cases of 3-Lie coalgebras of the type $C_e$ in the Theorem 3.2 are compatible with the 3-Lie algebra $L_{c_3}$. And we have isomorphisms of 3-Lie bialgebras

$$(L_{c_3}, C_e) :$$

$(1) \rightarrow (4) : f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = e_4;$
1) → (2), (3) → (5), (4) → (6) : \( f(e_1) = e_2, f(e_2) = e_1, f(e_3) = e_3, f(e_4) = e_4 \).

For every 3-Lie algebra isomorphism

\[
h : L_{c_3} \to L_{c_3} : h(e_j) = \sum_{k=1}^{4} a_{jk} e_k, a_{jk} \in F, j = 1, 2, 3, 4, a_{jk} \in F,
\]

we have \( \det \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = 1 \). If \( h \) is an isomorphism of the 3-Lie coalgebra (1) onto 3-Lie coalgebra (2), then \( a_{11} = a_{12} = a_{21} = a_{22} = 0 \). This contradicts to the derived algebra \( L^1_{c_3} = F e_1 + F e_2 \). The result follows.

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References


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