3-Lie Algebras and Cubic Matrices

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Abstract
The realization of n-Lie algebras is very important in the study of the structure of n-Lie algebras for n \( \geq 3 \). This paper considers the realizations of 3-Lie algebras by cubic matrices. First, the trace function \( tr_1 \) of cubic matrices is defined, and then the 3-ary Lie multiplication \( [\cdot, \cdot, \cdot]_{tr_1} \) on the vector space \( \Omega \) spanned by cubic matrices is constructed, and the structure of the 3-Lie algebra \( (\Omega, [\cdot, \cdot, \cdot]_{tr_1}) \) is investigated. It is proved that \( (\Omega, [\cdot, \cdot, \cdot]_{tr_1}) \) is a decomposable 3-Lie algebra, and there does not exist any metric on it.

Mathematics Subject Classification: 17B05, 17D99.

Keywords: 3-Lie algebra, cubic matrix, trace function.

1 Introduction

n-Lie algebras [1] are a kind of multiple algebraic systems appearing in many fields in mathematics and mathematical physics (cf [2, 3, 4]). In this paper, we pay our main attention to construct 3-Lie algebras. In papers [1, 5, 6, 7, 8] 3-Lie algebras are constructed by commutative associative algebras with derivations, \( \gamma \)-matrices, 2-dimensional extensions of metric Lie algebras, associative
algebras, Lie algebras and linear functions. Authors in paper [7] defined cubic matrices $A = (a_{ijk})$ for $1 \leq i, j, k \leq 2$, and R. Kerner [9] defined the ternary multiplication of three cubic matrices $(A \odot B \odot C)_{ijk} = \sum_{pqr} a_{pq} b_{qr} c_{rkj}$, and the symmetry properties of the ternary algebras are studied. The paper [10] defined five non-isomorphic 2-ary multiplication on the vector space $\Omega$ spanned by the cubic matrices $A$, and constructed non-isomorphic 3-Lie algebras. In this paper, we define the new trace function $tr_1$ of cubic matrices, and construct new 3-Lie algebra $(\Omega, [\cdot, \cdot, \cdot]_{tr_1})$, which is not isomorphic to the 3-Lie algebras in paper [10].

First we introduce some notions.

An $n$-Lie algebra $[1]$ is a vector space $V$ over a field $F$ equipped with an $n$-multilinear operation $[x_1, \cdots, x_n]$ satisfying

$$[x_1, \cdots, x_n] = \text{sign}(\sigma)[x_{\sigma(1)}, \cdots, x_{\sigma(n)}],$$

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^n [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]$$

for $x_1, \cdots, x_n, y_2, \cdots, y_n \in V$ and $\sigma \in S_n$, the permutation group on $n$ letters.

Denote by $[V_1, V_2, \cdots, V_n]$ the subspace of $V$ generated by all vectors $[x_1, x_2, \cdots, x_n]$, where $x_i \in V_i$, for $i = 1, 2, \cdots, n$. The subalgebra $V^1 = [V, V, \cdots, V]$ is called the derived algebra of $V$. If $V^1 = 0$, then $V$ is an abelian $n$-Lie algebra.

An ideal of an $n$-Lie algebra $V$ is a subspace $I$ such that $[I, V, \cdots, V] \subseteq I$. If $I$ satisfies $[I, I, V, \cdots, V] = 0$, then $I$ is called an abelian ideal.

The center of an $n$-Lie algebra $V$ is

$$Z(V) = \{x \in V \mid [x, V, \cdots, V] = 0\}.$$  

It is clear that $Z(V)$ is an abelian ideal of $V$.

A metric $n$-Lie algebra [11] is an $n$-Lie algebra $V$ that has a nondegenerate symmetric bilinear form $\rho$ on $V$, which is invariant,

$$\rho([x_1, \cdots, x_{n-1}, y_1], y_2) + \rho([x_1, \cdots, x_{n-1}, y_2], y_1) = 0, \text{ for all } x_i, y_j \in V.$$  

Such a bilinear form $\rho$ is called an invariant scalar product on $V$ or a metric on $V$. Note the $\rho$ is not necessarily positive definite.

Let $W$ be a subspace of a metric $n$-Lie algebra $V$. The orthogonal complement of $W$ is defined by

$$W^\perp = \{x \in V \mid \rho(w, x) = 0 \text{ for all } w \in W\}.$$  

If $W$ is an ideal, then $W^\perp$ is also an ideal and $(W^\perp)^\perp = W$, and $\dim W + \dim W^\perp = \dim V$.

In the following, we suppose that $F$ is a field of characteristic zero.
An $N$-order cubic matrix $A = (a_{ijk})$ (see [10]) over a field $F$ is an ordered object which the elements with 3 indices, and the element in the position $(i, j, k)$ is $(A)_{ijk} = a_{ijk}$, $a_{ijk} \in F$, $1 \leq i, j, k \leq N$. Denote the set of all cubic matrix over a field $F$ by $\Omega$. Then $\Omega$ is an $N^3$-dimensional vector space with

$$A + B = (a_{ijk} + b_{ijk}) \in \Omega, \quad \lambda A = (\lambda a_{ijk}) \in \Omega,$$

for $\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega$, $\lambda \in F$, that is, $(A + B)_{ijk} = a_{ijk} + b_{ijk}$, $(\lambda A)_{ijk} = \lambda a_{ijk}$.

Denote $E_{ijk}$ a cubic matrix with the element in the position $(i, j, k)$ is 1 and elsewhere are zero, that is, $E_{ijk} = (a_{lmn})$ with $a_{lmn} = \delta_l^i \delta_m^j \delta_n^k$, $1 \leq l, m, n \leq N$, and the cubic matrix $E_i = \sum_{j=1}^N E_{ijj}$, $1 \leq i \leq N$. Then $\{E_{ijk}, 1 \leq i, j, k \leq N\}$ is a basis of $\Omega$, and for every $A = (a_{ijk}) \in \Omega$, $A = \sum_{1 \leq i,j,k \leq N} a_{ijk}E_{ijk}$, $a_{ijk} \in F$.

Every cubic matrix $A$ can be written as the type of blocking form

$$A = (A_1^1, \cdots, A_N^1), \quad A_i^1 = (a_{ijk}), 1 \leq i \leq N, \quad (1.1)$$

where $A_i^1 = (a_{ijk}), 1 \leq i \leq N$ are usual $(N \times N)$-order matrices with the element $a_{ijk}$ at the position of the $j^{\text{th}}$-row and the $k^{\text{th}}$-column.

The paper [10] defined the multiplication $*_{11}$ on $\Omega$ and the trace function $\langle A \rangle_1$ of cubic matrices

$$\langle A *_{11} B \rangle_{ijk} = \sum_{p=1}^N a_{ijp} b_{ipk}, \quad \langle A \rangle_1 = \sum_{i,j,k=1}^N a_{ipp}. \quad (1.2)$$

Then for $A = (A_1^1, \cdots, A_N^1)$, $B = (B_1^1, \cdots, B_N^1) \in \Omega$,

$$A *_{11} B = (A_1^1B_1^1, \cdots, A_N^1B_N^1). \quad (1.3)$$

$(\Omega, *_{11})$ is an associative algebra and satisfies $\langle A *_{11} B \rangle_1 = \langle B *_{11} A \rangle_1$ for $A, B \in \Omega$. By [8], $\Omega$ is a 3-Lie algebra in the multiplication for $A, B, C \in \Omega$,

$$[A, B, C]_1 = \langle A \rangle_1 (B *_{11} C - C *_{11} B) + \langle B \rangle_1 (C *_{11} C - A *_{11} B) + \langle C \rangle_1 (A *_{11} C - B *_{11} B).$$

**Lemma 1.1** [10] The 3-Lie algebra $(\Omega, [\cdot, \cdot]_1)$ is an indecomposable 3-Lie algebra, and has a semi-direct product $\Omega = I + FE_1$, where $I$ is a maximal ideal of $(\Omega, [\cdot, \cdot]_1)$ with the codimension one and $E_1 = \sum_{j=1}^N E_{jjj}$.

## 2 3-Lie Algebra $(\Omega, [\cdot, \cdot]_{tr_1})$

For promoting the 3-Lie structure on $\Omega$, we define the new trace function $tr_1$ of cubic matrix. For $A = (A_1^1, \cdots, A_N^1) \in \Omega$, defines

$$tr_1(A) = tr(A_1) = \sum_{p=1}^N a_{1pp}. \quad (2.1)$$
Then we have the following result.

**Theorem 2.1.** $(\Omega, [\cdot, \cdot]_{tr_1})$ is a 3-Lie algebra in the multiplication
\[ [A, B, C]_{tr_1} = tr_1(A)(B *_{11} C - C *_{11} B) + tr_1(B)(C *_{11} A - A *_{11} B) + tr_1(C)(A *_{11} B - B *_{11} A). \] (2.2)

And the multiplication in the basis \{ $E_{ijk}, 1 \leq i, j, k \leq N$\} is as follows
\[ [E_{ijk}, E_{lmn}, E_{pqr}] = \delta_{jk} \delta_{ip} \delta_{qr} (\delta_{pq} E_{lmn} - \delta_{rm} E_{ijn}) + \delta_{in} \delta_{mn} \delta_{pr} (\delta_{ij} E_{pqk} - \delta_{kq} E_{pjr}) + \delta_{pt} \delta_{qr} \delta_{il} (\delta_{km} E_{ijn} - \delta_{jn} E_{imk}). \] (2.3)

**Proof** By Eqs. (1.3) and (2.1), $\forall A = (A^1, \ldots, A^N), B = (B^1, \ldots, B^N) \in \Omega$, $tr_1(A *_{11} B) = tr_1(B *_{11} A)$. Then by paper [8], $(\Omega, [\cdot, \cdot]_{tr_1})$ is a 3-Lie algebra in the multiplication (2.2). By Eqs. (1.2), (2.1) and (2.2), we obtain Eq. (2.3).

Now we study the structure of the 3-Lie algebra $(\Omega, [\cdot, \cdot]_{tr_1})$.

**Theorem 2.2** The 3-Lie algebra $(\Omega, [\cdot, \cdot]_{tr_1})$ can be decomposed into the direct sum of subalgebras
\[ \Omega = \Omega_1 + \cdots + \Omega_N, \]
where $\Omega_i = \{A = (A^1, \ldots, A^i, \ldots, A^N) \in \Omega \mid A = (0, \ldots, 0, A^i, 0, \ldots, 0)\}, 1 \leq i \leq N$. $\Omega_j, 2 \leq j \leq N$ are abelian subalgebras, but $\Omega_1$ is a non-abelian subalgebra. And $\Omega_1 + \Omega_{j_1} + \cdots + \Omega_{j_t}, 2 \leq j_1 < \cdots < j_t \leq N$, are non-abelian subalgebras.

**Proof** For every $2 \leq j \leq N$, and $A = (0, \ldots, 0, A^j, 0, \ldots, 0), B = (0, \ldots, 0, B^j, 0, \ldots, 0), C = (0, \ldots, 0, C^j, 0, \ldots, 0) \in \Omega_j$, by Theorem 2.1 and Eq (2.1), $tr_1(A) = tr_1(B) = tr_1(C) = 0$, therefore, $[A, B, C]_{tr_1} = 0$. Then $\Omega_j, 2 \leq j \leq N$ are abelian subalgebras.

Since there exists $A = (A^1, 0, \ldots, 0) \in \Omega_1$, such that $tr_1(A) = tr(A^1) \neq 0$, by Eqs (2.2), $\Omega_1$ is a non-abelian subalgebra. Similarly, for the cases $\Omega_1 + \Omega_{j_1} + \cdots + \Omega_{j_t}, 2 \leq j_1 < \cdots < j_t \leq N$. The result follows.

**Theorem 2.3** Let $(\Omega, [\cdot, \cdot]_{tr_1})$ be the 3-Lie algebra in Theorem 2.1. Then we have

1. The derived algebra $\Omega^1 = [\Omega, \Omega, \Omega]_{tr_1}$ has dimension $N^3 - N$, and
\[ \Omega^1 = \{A = (A^1, A^2, \ldots, A^N) \in \Omega \mid tr(A^i) = 0, 1 \leq i \leq N\}. \]

2. $\Omega_0 = \{A = (0, A^2, \ldots, A^N) \in \Omega\}$ is a non-abelian ideal of the 3-Lie algebra, but $\Omega_0$ is a maximal abelian subalgebra.

3. $\Omega = \{A = (A^1, A^2, \ldots, A^N) \in \Omega \mid tr(A^i) = 0\}$ is the maximal ideal of the 3-Lie algebra with codimension one.

4. The center $Z(\Omega)$ of the 3-Lie algebra $(\Omega, [\cdot, \cdot]_{tr_1})$ has dimension $N - 1$, and
\[ Z(\Omega) = \{A = (0, a_2 E, \ldots, a_N E) \in \Omega \mid a_2, \ldots, a_N \in F\}, \]
where $E$ is the $(N \times N)$-order unit matrix.
5) $(\Omega, [, , ]_{tr})$ is decomposable, and has a decomposition

$$\Omega = I \oplus Z(\Omega),$$

where $I = \Omega^1 + FE_1$ is an $(N^3 - N + 1)$-dimensional ideal of $\Omega$, $I \cap Z(\Omega) = 0$, where $E_1 = (E, 0, \cdots, 0)$.

**Proof** By Eqs. (1.3) and (2.2), for $A = (A_1, \cdots, A_N)$, $B = (B_1, \cdots, B_N)$, $C = (C_1, \cdots, C_N) \in \Omega$,

$$[A, B, C]_{tr} = tr_1(A)(B_1C_1 - C_1B_1, \cdots, B_NC_N - C_NB_N)$$

$$+ tr_1(B)(C_1A_1 - A_1C_1, \cdots, C_NA_N - A_NC_N)$$

$$+ tr_1(C)(A_1B_1 - B_1A_1, \cdots, A_NB_N - B_NA_N).$$

Since for $1 \leq i \leq N$,

$$tr(B_1C_1^i - C_1^iB_1) = tr(A_1^iB_1 - B_1^iA_1) = tr(C_1^iA_1 - A_1^iC_1) = 0,$$

the result 1) follows. The result 2) follows from the direct computation.

It is clear, $\dim \overline{\Omega} = N^3 - 1$. Follows from the result 1), $\overline{\Omega}$ contains the derived algebra $\Omega^1$. Therefore, $\overline{\Omega}$ is a maximal ideal of the 3-Lie algebra. The result 3) follows.

Let $A = (A_1, \cdots, A_N) \in \Omega$ be in the center of the 3-Lie algebra $(\Omega, [, , ]_{tr})$. Since for every $i \geq 2$,

$$[(E_{1i}, 0, \cdots, 0), (0, \cdots, 0, E_{ii}, 0, \cdots, 0), A]_{tr} = tr_1((E_{1i}, 0, \cdots, 0))[(0, \cdots, 0, E_{ii}, 0, \cdots, 0) *_{11} A - A *_{11} (0, \cdots, 0, E_{ii}, 0, \cdots, 0) + tr_1((E_{1i}, 0, \cdots, 0) *_{11} (0, \cdots, 0, E_{ii}, 0, \cdots, 0) - (0, \cdots, 0, E_{ii}, 0, \cdots, 0) *_{11} (E_{1i}, 0, \cdots, 0))] = (0, \cdots, 0, E_{ii}A_1 - A_1E_{ii}, 0, \cdots, 0) = 0,$$

we obtain $A_1 = a_i E$, where $E_{ii}$, $1 \leq i \leq N$ are $(N \times N)$-order matrices with 1 at the position $i$-th row and $i$-th-column and others are zero, $E = \sum_{i=1}^{N} E_{ii}$ is the $(N \times N)$-order unit matrix. Then, the cubic matrix $A$ has the form $A = (A_1, a_2 E, \cdots, a_N E)$. For $1 \leq i \neq j \neq k, i \neq k \leq N$, since

$$[A, (E_{ij}, 0, \cdots, 0), (E_{jk}, 0, \cdots, 0)] = [(A_1, E, \cdots, E), (E_{ij}, 0, \cdots, 0), (E_{jk}, 0, \cdots, 0)] = tr(A_1)(E_{ik}, 0, \cdots, 0) = (0, \cdots, 0),$$

we have $tr(A_1) = 0$. For every $1 \leq i \neq j \leq N$,

$$[A, (E_{ij}, 0, \cdots, 0), (E_{1i}, 0, \cdots, 0)] = (A_1E_{ij} - E_{ij}A_1, 0, \cdots, 0) = ((a_{ii} - a_{jj})E_{ij}, 0, \cdots, 0) = 0,$$

and $A_1 = a_i E$. Thanks to $tr(A_1) = Na = 0, A_1 = 0$. The result 4) follows.

Clearly, $I = \Omega^1 + FE_1$ is an ideal of the 3-Lie algebra $(\Omega, [, , ]_{tr})$ since $I \supset \Omega^1$. By result 1) and result 4), $I \cap Z(\Omega) = 0$, $\dim I + \dim Z(\Omega) = N^3$, and $[I, Z(\Omega), \Omega]_{tr} = 0$. Therefore, $\Omega = I \oplus Z(\Omega)$. The result 5) follows.

**Theorem 2.4** There does not exist metric structure on the 3-Lie algebra $(\Omega, [, , ]_{tr})$. 


Proof. If $\rho$ is a metric on the 3-Lie algebra $(\Omega, [\cdot, \cdot]_{tr})$. Then $\rho$ is a nondegenerate symmetric bilinear form on $\Omega$ satisfying
\[
\rho([A, B, C]_{tr1}, D) + \rho(C, [A, B, D]_{tr1}) = 0, \forall A, B, C, D \in \Omega.
\]
Thanks to Lemma 2.3 in paper [11], $Z(\Omega) \perp = \Omega^1$. Then we have
\[
\dim \Omega^1 + \dim Z(\Omega) = N^3 - N + (N - 1) = N^3 - 1 = N^3.
\]
Contradiction. It follows the result.

ACKNOWLEDGEMENT. This research is supported by NSF of China (11371245).

References


Received: February, 2014