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weak ϕ-contraction on partial metric spaces and existence of fixed points in partially ordered sets

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Abstract

In this manuscript, the notion of weak ϕ-contraction is considered on partial metric space. It is shown that a self mapping T on a complete partial metric space X has a fixed point if they satisfied weak ϕ-contraction.

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1 Introduction and Preliminaries

In 1992, Matthews [1, 2] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which $d(x, x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (See e.g. [3, 4, 5, 6, 16])

A partial metric space (See e.g.[1, 2]) is a pair $(X, p : S \times S \to \mathbb{R}^+)$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) such that

(PM1) $p(x, y) = p(y, x)$ (symmetry)

(PM2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)

(PM3) $p(x, x) \leq p(x, y)$ (small self-distances)

(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity) for all $x, y, z \in X$. We use the abbreviation PMS for the partial metric space (X, p).

Notice that for a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

is a (usual) metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family open of p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$

Definition 1.1 *(See e.g.[1, 2, 6])*

(i) A sequence $\{x_n\}$ in a PMS (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$,

(ii) a sequence $\{x_n\}$ in a PMS (X, p) is called a Cauchy if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite),

(iii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 1.2 *(See e.g.[1, 2, 6])*

(A) A sequence $\{x_n\}$ is Cauchy in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, d_p),

(B) A PMS (X, p) is complete if and only if a metric space (X, d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) \tag{2}$$

Boyd and Wong [7] introduced the notion of Φ-contraction: a self mapping T on a metric space X is called Φ-contraction if there exists an upper semi-continuous function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq \Phi(d(x, y)) \quad \text{for all } x, y \in X.$$

Alber and Guerre-Delabriere [8], generalized the notion of Φ-contraction by defining the notion of weak ϕ-contraction for Hilbert spaces: A self mapping

T on a metric space X is called weak ϕ-contraction if $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing map with $\phi(0) = 0$ and

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \text{ for all } x, y \in X.$$

They also proved the existence of fixed points in Hilbert spaces. I one replaces Hilbert spaces with an arbitrary Banach spaces [8] still we have fixed points(See e.g. [9]). We should note that for a lower semi-continuous mapping ϕ, the function $\Phi(u) = u - \phi(u)$ coincides with Boyd and Wong types.

In fixed point theory, Φ-contraction and weak ϕ-contraction have been studied by many authors, (See e.g.,[10, 11, 12, 13, 14], also [15]). In this manuscript, by using weak ϕ-contraction on a complete partial metric space we obtain a unique fixed point.

2 Main Results

Definition 2.1 *(cf. [14]) Let $(X, \preceq)$ be a partially ordered set and (X, p) a complete partial metric space. An operator $T : X \to X$ is called a weak ϕ-contraction if there exists a continuous, non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$, such that*

$$p(Tx, Ty) \leq p(x, y) - \phi(p(x, y)) \tag{3}$$

for any $x,\ y \in X$ with $x \preceq y$.

Theorem 2.2 *Let $(X, \preceq)$ be a partially ordered set and (X, p) a complete partial metric space. Suppose that $T : X \to X$ is nondecreasing, continuous and weak ϕ-contraction. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, T has a unique fixed point.*

Proof. Let $x_0 \in X$ and set $x_{n+1} = Tx_n$. Notice that, if $x_n = x_{n+1}$ for any $n \geq 0$, then obviously T has a fixed point. Thus, suppose $x_n \neq x_{n+1}$ for any $n \geq 0$. Since $x_0 \preceq Tx_0$, then

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{4}$$

Due to (3), we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq p(x_n, x_{n+1}) - \phi(p(x_n, x_{n+1})).$$

Define $t_n = p(x_n, x_{n+1})$.
Then one can obtain

$$t_{n+1} \leq t_n - \phi(t_n) \leq t_n \tag{5}$$

which implies that $\{t_n\}$ is a non-negative non-increasing sequence. Hence, $\{t_n\}$ converges to L where $L \geq 0$. So there are two cases: $L > 0$ or $L = 0$. Assume that $L > 0$. Regarding that ϕ is non-decreasing, we get $0 < \phi(L) \leq \phi(t_n)$. Due to (5), we have $t_{n+1} \leq t_n - \phi(t_n) \leq t_n - \phi(L)$ and so

$$t_{n+2} \leq t_{n+1} - \phi(t_{n+1}) \leq t_n - \phi(t_n) - \phi(t_{n+1}) \leq t_n - 2\phi(L).$$

Inductively we obtain $t_{n+k} \leq t_n - k\phi(L)$ which is a contradiction for large enough $k \in \mathbb{N}$. Hence we have $L = 0$. Thus, we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, p). For this purpose, define $s_n = \sup\{p(x_i, x_j) : i, j \geq n\}$. It is clear that the sequence $\{s_n\}$ is decreasing. If $\lim_{n\to\infty} s_n = 0$, then $\{x_n\}$ is a Cauchy sequence. So consider the other case: Suppose $\lim_{n\to\infty} s_n = s > 0$. One can choose ε small enough (e.g. $\varepsilon < \frac{s}{16}$) and a natural number N such that

$$p(x_n, x_{n+1}) < \varepsilon, \quad \text{and} \quad s_n < s + \varepsilon, \quad \text{for all} \quad n \geq N. \tag{6}$$

Regarding the definition of s_{N+1}, there exist $m, n \geq N + 1$ such that

$$s - \varepsilon < s_n - \varepsilon < p(x_m, x_n). \tag{7}$$

By triangle inequality we observe that

$$p(x_n, x_m) \leq p(x_n, x_{n-1}) + p(x_{n-1}, x_m) - p(x_{n-1}, x_{n-1}) \tag{8}$$

$$p(x_n, x_m) \leq p(x_n, x_{m-1}) + p(x_{m-1}, x_m) - p(x_{m-1}, x_{m-1}) \tag{9}$$

$$p(x_{n-1}, x_m) \leq p(x_{n-1}, x_{m-1}) + p(x_{m-1}, x_m) - p(x_{m-1}, x_{m-1}) \tag{10}$$

Due to (7) and (6) the expression (8) and (9) yield that

$$s - 2\varepsilon < p(x_{n-1}, x_m), \quad \text{and} \quad s - 2\varepsilon < p(x_n, x_{m-1}). \tag{11}$$

Combining (10) and (11), we get that

$$s - 3\varepsilon < p(x_{n-1}, x_{m-1}). \tag{12}$$

Thus,

$$\begin{aligned} p(x_n, x_m) &= p(Tx_{n-1}, Tx_{m-1}) \leq p(x_{n-1}, x_{m-1}) - \phi(p(x_{n-1}, x_{m-1})) \\ &\leq p(x_{n-1}, x_{m-1}) - \phi(s) \end{aligned} \tag{13}$$

Regarding (7) and (12), the expression (13) implies that $s_{N+1} < s_N - \phi(s)$ for small enough ε. It is impossible. Hence $s = 0$. Notice that

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_{n-1}, x_{n-1}) - \phi(p(x_{m-1}, x_{m-1})) \tag{14}$$

Since $s = 0$, then $d_p(x_n, x_m) \longrightarrow 0$ Therefore, the sequence $\{x_n\}$ is Cauchy in (X, d_p). Since (X, p) is complete, by Lemma 2 (X, d_p) is complete. and the sequence $\{x_n\}$ is convergent in X, say $z \in X$. Again by Lemma 2,

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) \tag{15}$$

Since $s = 0$, then by (15) we have $p(z, z) = 0$. We assert that $Tz = z$. Due to (PM4), we have

$$\begin{aligned} p(Tz, z) &\leq p(Tz, Tx_n) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_n) - \phi(p(z, x_n)) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}) \end{aligned} \tag{16}$$

Letting $n \to \infty$ and regarding the continuity of ϕ, then (16) yields that $p(Tz, z) \leq 0$. Hence $Tz = z$.

Now we show z is unique fixed point of T. Assume the contrary, that is, there exists $w \in X$ such that $z \neq w$ and $w = Tw$.

$$p(z, w) = p(Tz, Tw) \leq p(z, w) - \phi(p(z, w))$$

which is a contradiction. Thus z is a unique fixed point of T. ∎

Theorem 2.3 *Let $(X, \preceq)$ be a partially ordered set and (X, p) a complete partial metric space. Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. Suppose also that $T : X \to X$ is nondecreasing and satisfying*

$$p(Tx, Ty) \leq p(x, y) - \phi(p(x, y)) \tag{17}$$

for any $x,\ y \in X$ with $x \prec$ (that is, $x \preceq y$ and $x \neq y$). Moreover the following condition is hold:

$$\textit{If } \{x_n\} \subset X \textit{ is a increasing sequence with } x_n \to z, \textit{ then } x_n \prec z, \ \forall n. \tag{18}$$

If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, T has a fixed point.

Proof. As in the proof of Theorem 2.2, take $x_0 \in X$ and set $x_{n+1} = Tx_n$. If $x_n = x_{n+1}$ for any $n \geq 0$, then obviously T has a fixed point. Thus, suppose $x_n \neq x_{n+1}$ for any $n \geq 0$. Since $x_0 \preceq Tx_0$, then

$$x_0 \prec x_1 \prec \cdots \prec x_n \preceq x_{n+1} \prec \cdots \tag{19}$$

As in the proof of Theorem 2.2, we observe that the sequence $\{x_n\}$ is Cauchy and thus it converges to $z \in X$. Hence, we have (as in the proof of Theorem 2.2)

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0 \tag{20}$$

We assert that $Tz = z$. Due to (18) and (PM4), we have

$$\begin{aligned} p(Tz, z) &\leq p(Tz, Tx_n) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_n) - \phi(p(z, x_n)) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}) \end{aligned} \tag{21}$$

Letting $n \to \infty$ and regarding the continuity of ϕ, then (16) yields that $p(Tz, z) \leq 0$. Hence $Tz = z$. ■

If we take $\Phi(t) = t - \phi(t)$, then one can easily see that Φ satisfies all conditions of the main theorem of [6]. So we can state some results of [6] as a corollary of our theorem.

Corollary 2.4 *(See [6]) Let $(X, \preceq)$ be a partially ordered set and (X, p) a complete partial metric space. Suppose $T : X \to X$ be a self mapping such that*

$$p(Tx, Ty) \leq \Phi(p(x, y)), \quad \textit{for all } x, y \in X, \ \textit{with } x \preceq y$$

where $\Phi(t) : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing function such that $\phi(t) < t$ for each $t > 0$. Then T has a unique fixed point.

If we take $\Phi(t) = kt$ we get Banach contraction principle for PMS.

Corollary 2.5 *(See [2, 4, 6]) Let $(X, \preceq)$ be a partially ordered set and (X, p) a complete partial metric space. Suppose $T : X \to X$ be a self mapping such that*

$$p(Tx, Ty) \leq kp(x, y), \quad \textit{for all } x, y \in X, \ \textit{with } x \preceq y$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Example 2.6 *Let $X = I\!R^+$ and $p(x, y) = \max\{x, y\}$ then (X, p) is a PMS (See e.g. [6].) Suppose $T : X \to X$ such that $Tx = \begin{cases} \frac{x^2}{1+x} & \textit{for all } x \in [0, 1] \\ 2x & \textit{for all } x \in (1, \infty) \end{cases}$ and $\phi(t) : [0, \infty) \to [0, \infty)$ such that $\phi(t) = \frac{t}{1+t}$. It is clear that T is nondecrasing. For $x \prec y$ we have*

$$p(Tx, Ty) = \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} = \frac{x^2}{1+x} \leq x - \frac{x}{1+x} = \frac{x^2}{1+x}$$

Thus, it satisfies all conditions of the Theorem 2.3. Notice also that, for choosing $\Phi(t) = t - \phi(t) = \frac{t^2}{1+t}$, all conditions of Theorem 1 of [6] and guarantee that T has a unique fixed point, indeed $x = 0$ is the required point.

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