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Wavelet Based Methods for Numerical Solutions of Two Dimensional Integral Equations

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Abstract

Integral equations are useful in many branches of mathematics and science as well. We begin with a brief summary of solving different kinds of one dimensional integral equations, namely, Fredholm Integral equation of the first and the second kind, Volterra integral equation of the second kind and Fredholm-Volterra integral equation as well as the discussions of singular and nonlinear integral equations. We will also discuss solving two-dimensional integral equations. There are many different methods of solving integral equations. Wavelet based methods are of particular interest. The localization property, robustness and other features of wavelets are essential to solving integral equations efficiently. We will present a wavelet based method together with several convergence results of the method. A few examples will also be presented. Some of these examples have been tested by others. We will compare the results with other methods.

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1 Introduction

The study of systems of linear equations is a fundamental subject in many areas of pure and applied mathematics. Since an integral can be regarded as a limit of finite sum, an integral equation is an infinite dimensional counterpart of the finite dimensional linear system. Therefore, many real world problems can be modelled by using integral equations. In this paper, we will summarize some results on one dimensional integral equations we obtained previously and discuss solving two-dimensional integral equations by using coiflets. We begin with a general discussion of different methods solving various types of integral equations, namely, Fredholm Integral equation of the first and the second kind, Volterra integral equation of the second kind and Fredholm-Volterra integral equation. We then provide a brief account of wavelet theory. The useful feature of localization for wavelets results from related recursive basis constructions, which are obtained from multiresolution analysis [3, 4]. In the subsequent sections, we present a wavelet based method in solving two dimension integral equations by using coiflets followed by proving theorems on error analysis which give rise to convergence results of our method. We also present some numerical examples that validate our method.

1.1 One Dimensional Integral Equations

Many different methods have been applied for solving Fredholm integral equations numerically, such as adaptive multiscale moment, collocation method, expansion method, etc [2, 6, 10]. In a series of papers [7-9], we have used wavelet based methods in solving the following one dimensional integral equations.

Fredholm Integral equation of the second kind [7]:

$$u(x) = f(x) + \int_a^b k(x,t)u(t)dt, t \in [a,b], \tag{1}$$

Fredholm Integral equation of the first kind [9]:

$$u(x) = f(x) + \int_a^b k(x,t)u(t)dt, t \in [a,b], \tag{2}$$

Volterra Integral equation of the second kind [8]:

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt, t \in [a,X], a \le x \le X \le b, \tag{3}$$

Fredholm-Volterra integral equation [8]:

$$u(x) = f(x) + \int_a^b k_1(x,t)u(t)dt + \int_a^x k_2(x,t)u(t)dt, a \le x \le X \le b. \tag{4}$$

We have provided error analysis for each of the above solutions and obtained better solutions than other methods [7-9]. Some nonlinear or singular integral equations may also be solved in a similar fashion.

1.2 Two Dimentional Integral Equations

We next present a wavelet-based method in solving two-dimensional Fredholm integral equations of the first kind of the form:

$$f(x,y) = \int_0^1 \int_0^1 k(x,y,s,t)u(s,t)dsdt, \tag{5}$$

and two-dimensional Fredholm integral equation of the second kind as follows:

$$u(x,y) = f(x,y) + \int_0^1 \int_0^1 k(x,y,s,t)u(s,t)dsdt, \tag{6}$$

where the functions $f(x,y), k(x,y,s,t)$ are given, k is called the kernel function. To solve equations (5) and (6), we mean to find the unknown function $u(x,y)$ by using numerical methods.

We extend the one dimensional method to solve two-dimensional integral equations.

1.3 Coiflets and Wavelet Interpolation

This section provides a brief description of wavelet function, coiflet and wavelet interpolation. The wavelet function is defined by building the set of scaling functions φ. We consider the sequence $\{a_p, p \in Z\}$, and a family of functions with dilatation and translation

$$\varphi(x) = \sum_p a_p \varphi(2^j x - p) = \sum_p a_p \varphi_{j,p}(x). \tag{7}$$

A nested of subspaces $\{V_j, j \in Z\}$ of $L^2(\Re)$ is defined such that,

$$V_j = \overline{Span\{\varphi_{j,p}(x), p \in Z\}}, \quad j \in Z, \tag{8}$$

which means that for any function $f(x) \in V_j$, it can be expressed as

$$f(x) = \sum_p \alpha_p \varphi_{j,p}(x). \tag{9}$$

If the bases of subspace are orthogonal at the same level, then for the function can be approximated as follows:

$$f(x) = \sum_p < f, \varphi_{j,p} > \varphi_{j,p}(x), \tag{10}$$

where

$$< f, \varphi_{j,p} >= \int_{-\infty}^{\infty} f(x)\varphi_{j,p}(x)dx. \tag{11}$$

In fact, if the nested sequence of the subspaces $\{V_j, j \in Z\}$ has the following properties then it is called a multiresolution analysis (MRA):

$$V_j \subset V_{j+1}, \tag{12}$$

$$f(x) = V_j \Leftrightarrow f(2x) \in V_{j+1}, \tag{13}$$

$$\cap_{j \in Z} V_j = \{0\}, \tag{14}$$

$$\overline{\cup_{j \in Z} V_j} = L^2(\Re), \tag{15}$$

there exists a function $\varphi \in V_0$ such that $\{\varphi(x-k), k \in Z\}$ is an orthogonal basis for V_0 .

The wavelet function $\psi(x)$ is constructed in the orthogonal complement of each subspace V_j in V_{j+1} which is denoted by W_j. In other words, $V_{j+1} = V_j \oplus W_j$. Therefore,

$$V_j \rightarrow \begin{cases} 0 & \text{as } j \rightarrow -\infty \\ L^2(\Re) & \text{as } j \rightarrow \infty. \end{cases}$$

Hence $V_{j+1} = V_j \oplus W_j = \oplus_{j=-\infty}^{\infty} W_j$ and $L_2(\Re) = \oplus_{j=-\infty}^{\infty} W_j$, the set $\{\psi_{j,p}(x) = \psi(2^j x - p), p \in Z\}$ from a basis for W_j, and

$$\psi(x) = \sum_p b_p \varphi_{j,p}(x) \ for \ \ some \ \ b_p. \tag{16}$$

The orthogonality of W_j on V_j means that any member of V_j is orthogonal to the members of W_j , that is,

$$< \phi_{j,p}, \psi_{j,k} >= \int \phi_{j,p}(x) \psi_{j,k}(x) dx = \delta_{p,k}. \tag{17}$$

In fact, scaling function and wavelet have more properties:

$$\int \varphi(x) dx = 1, \tag{18}$$

$$\int x \varphi(x) dx = \frac{1}{2} \sum_p p a_p = c, \tag{19}$$

$$\int x^r \psi(x) dx = 0, r = 0, \cdots, N-1, \tag{20}$$

where N is related to the compact support of $\varphi(x)$ and $\psi(x)$.

Definition 1.1 *The Coifman wavelet system (coiflet) of order L is an orthogonal multiresolution wavelet system with*

$$\int x^k \varphi(x)dx = 0, k = 1, \cdots, L-1 \tag{21}$$

$$\int x^k \psi(x)dx = 0, k = 0, 1, \cdots, L-1. \tag{22}$$

We recall an interpolation theorem in $\Re^2$ and $\Re^n$ as follows [1].

Theorem 1.2 *Assume the function $f(x) \in C^k(\bar{\Omega})$ wnere Ω is a bounded open set in R^2, $k \geq N \geq 2$. Let, for $j \in Z$*

$$f^j(x,y) = \frac{1}{2^j}\sum_{p,q} f(\frac{p+c}{2^j}, \frac{q+c}{2^j})\varphi_{j,p}(x)\varphi_{j,q}(y), \ (x,y) \in \Omega, \tag{23}$$

where the index $p, q \in \Lambda = \{(p,q)|(supp(\varphi_{j,p})\otimes supp(\varphi_{j,q})) \cap \Omega \neq \emptyset\}$.
In addition, the moment M_l satisfy

$$M_l = \int x^l \varphi(x)dx = (c)^l, \quad l = 1, 2, \cdots, N-1, \tag{24}$$

then

$$\|f - f^j\|_{L^2(\Omega)} \ \leq C\|f^{(N)}\|_\infty (\frac{1}{2^j})^{N-1}, \tag{25}$$

where C is a constant depending only on N, diameter of Ω and

$$\|f^{(N)}\|_\infty = max_{(x,y)\in\Omega}|\frac{\partial^N f}{\partial x^m \partial y^{N-m}}(x,y)|. \tag{26}$$

2 Using Coiflets for Solving Two-dimensional Integral Equations

In this section we use coiflet to solve two-dimensional Fredholm integral equations (5) and (6), where we will explain the method in terms of matrix notations.

2.1 Two-Dimensional Fredholm Integral equation of the first kind

The unknown function $u(s,t)$ in equation (5) can be expressed in term of scaling functions in the subspace V_j such that,

$$u^j(x,y) = \sum_p \sum_q a_{p,q}\varphi_{j,p}(x)\varphi_{j,q}(y). \tag{27}$$

By substituting equation (27) in (5), we will have the following system

$$f(x,y) = \int_0^1 \int_0^1 k(x,y,s,t) \sum_p \sum_q a_{p,q}\varphi_{j,p}(s)\varphi_{j,q}(t)dsdt \tag{28}$$

The system (28) can be written in form of matrix equation by using the notation

$$A_{p,q}(x,y) = \int_0^1 \int_0^1 k(x,y,s,t)\varphi_{j,p}(s)\varphi_{j,q}(t)dsdt \tag{29}$$

Then equation (28) becomes

$$f(x,y) = \sum_p \sum_q a_{p,q}A_{p,q}(x,y) \tag{30}$$

By providing appropriate collocation points in [0,1]×[0,1] for equation (28) the unknowns $a_{p.q}$ can be found. Equation (30) is equivalent to the following matrix equation:

$$\vec{f} = \vec{a}.A \tag{31}$$

where,

$$\vec{f} = (f(x_1,y_1), \cdots, f(x_n,y_n))^T \tag{32}$$

$$\vec{a} = (a_{1,1}, a_{1,2}, \cdots, a_{n,n})^T \tag{33}$$

$$A = \begin{pmatrix} A_{1,1}(x_1,y_1) & A_{1,2}(x_1,y_1) & \ldots & A_{n,n}(x_1,y_1) \\ A_{1,1}(x_1,y_2) & A_{1,2}(x_1,y_2) & \ldots & A_{n,n}(x_1,y_2) \\ \vdots & \vdots & & \vdots \\ A_{1,1}(x_n,y_n) & \cdot & \ldots & A_{n,n}(x_n,y_n) \end{pmatrix} \tag{34}$$

The solution of the system is

$$\vec{a} = A^{-1}.\vec{f} \tag{35}$$

This gives rise to coefficients in (27) and we obtain a numerical solution to equation (5).

2.2 Two-Dimensional Fredholm Integral Equation of the Second Kind

To solve the two-dimensional Fredholm integral equation of the second kind (6), we use a similar algorithm as shown in 2.1. The unknown function $u(x, y)$ in equation (6) can be approximated by converting equation (6) in the subspace V_j . One then has the system of linear equations;

$$A(B - c) = f \tag{36}$$

where,

$$A = [a_{pq}], \tag{37}$$

$$B = [\varphi_{j,p}(x)\varphi_{j,q}(y)], \tag{38}$$

$$C = \int_0^1 \int_0^1 k(x_i, y_l, s, t)\varphi_{j,p}(s)\varphi_{j,q}(t)dsdt, \quad and \quad f = [f(x_i, y_l)]. \tag{39}$$

Then, the unknown coefficients a_{pq} can be evaluated by solving the linear system (36) such that;

$$A = f(B - C)^{-1} \tag{40}$$

This gives rise to a numerical solution for equation (6).

3 Error Analysis

In this section we discuss the convergence rate of our method in solving two-dimensional Fredholm integral equations. We present the necessary conditions for the convergence of our solution and indicate the benefits of the orthogonally of the coiflets.

Theorem 3.1 *Suppose that equation (5) is approximated by using (27) and the unknowns $a_{p,q}$ are evaluated by solving the matrix equation (31). Suppose that for each given point (x_i, y_r) the kernel function $k(x_i, y_r, s, t)$ is continuous in the square [0, 1] × [0, 1] and the function is positive. Moreover, suppose that the unknown function $u(x, y) \in C^k([0,1]\times[0,1])$, $k \geq N \geq 2$. If the error function at each point is*

$$e(x,y) = u^j(x,y) - u(x,y) \tag{41}$$

Then

$$\|e\| \leq con(\frac{1}{2})^{2j}. \tag{42}$$

Where con is a constant depending on the kernel function.

Proof: At the point (x_i, y_r) , plug-in the error function in the integral equation (5), we have the equation

$$\int_0^1 \int_0^1 k(x_i, y_r, s, t)e(s,t)dsdt \tag{43}$$

$$= \int_0^1 \int_0^1 k(x_i, y_r, s, t)(\sum_p \sum_q a_{pq}\varphi_{j,p}(s)\varphi_{j,q}(t) - u(s,t)dsdt).$$

We assumed that the kernel function is positive at each point in the square [0,1]×[0,1] then

$$\| \int_0^1 \int_0^1 k(x_i, y_r, s, t)e(s,t)dsdt\| \geq m_{ir}\| \int_0^1 \int_0^1 e(s,t)dsdt\| = m_{ir}\|e\| \tag{44}$$

where m_{ir} =$\min_{0\leq s,t\leq 1} k(s_i, y_r, s, t)$. Using (43) and (44) we have the following,

$$\|e\| \leq \frac{1}{m_{ir}}\| \int_0^1 \int_0^1 k(x_i, y_r, s, t)\|\|\int_0^1 \int_0^1 (\sum_p \sum_q a_{pq}\varphi j, p(s)\varphi_{j,q}(t) - u(s,t))dsdt\|$$

$$= c_{ir}\| \int_0^1 \int_0^1 (\sum_p \sum_q a_{pq}\varphi_{j,p}(s)\varphi_{j,q}(t) - u(s,t))dsdt\|$$

such that $c_{ir} = \frac{1}{m_{ir}}\| \int_0^1 \int_0^1 k(x_i, y_r s, t)dsdt\|$.

Adding and subtracting equation (23) inside the integral of the above equation, we get

$$\|e\| \leq c_{ir}\| \int_0^1 \int_0^1 (\sum_p \sum_q a_{p,q}\varphi_{j,p}(s)\varphi_{j,q}(t) - u(s,t) + u^j(s,t) - u^j(s,t))dsdt\|$$

$$\leq c_{ir}(\| \int_0^1 \int_0^1 (\sum_p \sum_q a_{p,q}\varphi_{j,p}(s)\varphi_{j,p}(t) - u^j(s,t))dsdt\| + \| \int_0^1 \int_0^1 (u^j(s,t) - u(s,t))dsdt\|)$$

$$\leq c_{ir}(\| \int_0^1 \int_0^1 (\sum_p \sum_q a_{p,q}\varphi_{j,p}(s)\varphi_{j,p}(t) - \sum_p \sum_q u(\frac{p}{2^j}, \frac{q}{2^j})\varphi_{j,p}(s)\varphi_{j,q}(t)))dsdt\|$$

$$+c_{ir}\| \int_0^1 \int_0^1 (u^j(s,t) - u(s,t))dsdt\|)$$

$$= c_{ir}(\sum_p \sum_q (a_{pq} - u(\frac{p}{2^j}, \frac{q}{2^j})) \int_0^1 \int_0^1 \varphi_{j,p}\varphi_{j,q}(t)dsdt) + c_{ir} \int_0^1 \int_0^1 (u^j(s,t) - u(s,t))dsdt.$$

By using the orthonormality on the basis functions $\{\varphi_{i,j}\}$, using Theorem 1.2 and by assuming N is large enough. Then we have

$$\|e\| = \alpha(\frac{1}{2})^{2j} + \beta\|u^{(N)}(s,t)\|(\frac{1}{2})^N = con(\frac{1}{2})^{2j}, \tag{45}$$

for some constants α , β and con.

Theorem 3.2 *If equation (6) is approximated by using (23) and the unknown coefficients are evaluated by solving the matrix equation (36). Moreover, suppose that for each point $(x_i, y_i) \in$ [0,1]x[0,1], the kernel function $k(x_i, y_i, s, t)$ is continuous on the square [0,1]×[0,1] and the solution $u(x,y) \in C^k$ [0,1]×[0,1], $k \geq N \geq 2$, then the error function at the point $(x_i, y_l) \in$[0,1]×[0,1] is bounded by*

$$\|e(x_i, y_l)\| = \|u^j(x_i, y_l) - u(x_i, y_l)\| \leq c\big(\frac{1}{2}\big)^{2j} \tag{46}$$

where c is a constant depending on the kernel function.

Proof:

$$\|e(x_i, y_l)\| \tag{47}$$

$$= \| \sum_q \sum_p a_{pq}\varphi_{j,p}(t)\varphi_{j,q}(s) - u(s,t)\|$$

$$= \| \int_0^1 \int_0^1 k(s_i, y_l, s, t)(\sum_q \sum_p a_{pq}\varphi_{j,p}(t)\varphi_{j,q}(s) - u(s,t))dsdt\|$$

$$\leq \| \int_0^1 \int_0^1 k(x_i, y_l, s, t)dsdt\| \| \int_0^1 \int_0^1 \sum_p a_{pq}\varphi_{j,p}(t)\varphi_{j,q}(s) - u(s,t))dsdt\|$$

$$= c_1 \| \int_0^1 \int_0^1 (\sum_p \sum_q a_{pq}\varphi_{j,p}(t)\varphi_{j,q}(s) - u(s,t))dsdt \|$$

where $c_1 = \| \int_0^1 \int_0^1 k(x_i, y_l, s, t)dsdt \|$.

Adding and subtractin the value $\sum_p \sum_q u(\frac{p}{2^j}, \frac{q}{2^j})\varphi_{j,p}(t)\varphi_{j,q}(s)$ to the above equation, we get:

$$\|e\| \leq \| \int_0^1 \int_0^1 (\sum_p \sum_q a_{pq}\varphi_{j,p}(t)\varphi_{j,q}(s) - \sum_p \sum_q u(\frac{p}{2^j}, \frac{q}{2^j})\varphi_{j,p}(t)\varphi_{j,q}(s))dsdt \| \tag{48}$$

$$+ \| \int_0^1 \int_0^1 (\sum_p \sum_q u(\frac{p}{2^j}, \frac{q}{2^j})\varphi_{j,p}(t)\varphi_{j,q}(s) - u(s,t))dsdt \|$$

By using Theorem 1.2 and the orthonormality of the scaling functions $\{\varphi(x)\}$, equation (48) becomes $\|e\| \leq c(\frac{1}{2})^{2j}$ for some constant c.

4 Integral Equations for Image Processing

In the study of image analysis, some images can be described by an integral equation, for example, it is shown in [5], images can be modelled by integral equations, more precisely, they can be modelled as:

$$Kv = f, \tag{49}$$

where

f is the observed image with noise, which are the pixel values of the image.

$v : \Omega \subset \Re \to \Re$ is the original image.

K is a two-dimensional Fredholm integral equation operator and is called the blurring operator which is defined by

$$Kv = \int_\Omega k(x, x')v(x')dx' \tag{50}$$

and k is the kernel function which depends on the image. In the next example, we show some images which are modelled by equation (49) with different kernel functions.

Example 1:

1) In remote sensing and astronomical images, we can use the following kernel function for equation (49).

$$k(x, y, x, y) = \frac{1}{\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\left(\frac{x - x'}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{y - y'}{\sigma_2}\right)^2 \right) \tag{51}$$

2) The following kernel function

$$k(x,y,x',y') = \frac{\sin(\pi w(x-x'))}{\pi w(x-x')} \cdot \frac{\sin(\pi w(y-y'))}{\pi w(y-y'))} \tag{52}$$

is used in the modelling for confocal microscopy, where represents the width of an aperture of the collector lens in confocal microscopy.
3) An out-of-focus image has the kernel

$$k(x,y,x',y') = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-x')^2+(y-y')^2}{2\sigma^2}\right) \tag{53}$$

where σ is the degree of accuracy and clearness of the image in the system.

5 Numerical Examples

Example 2:
Consider the two-dimensional integral equation (5) with

$$k(x,y,s,t) = xye^{s+t} \quad and \quad f(x,y) = xy, \tag{54}$$

and the exact solution $u(x,y) = e^{-x-y}$ is approximated by using coiflets of order 5 with $j = -10$. The numerical results are shown in Table 1.

Table 1: Results of Example 2

(x_i, y_i)	Approximation Values	Absolute Errors
(0.1,0.1)	0.81873	2.61929E-7
(0.2,0.2)	0.67032	2.14431E-7
(0.3,0.3)	0.548811	1.74984E-7
(0.4,0.4)	0.449329	1.43231E-7
(0.5,0.5)	0.367879	1.17272E-7
(0.6,0.6)	0.301185	9.60028-7
(0.7,0.7)	0.246589	7.85926E-5
(0.8,0.8)	0.20189	6.43397E-6
(0.9,0.9)	0.165294	5.26715E-6
(1.0,1.0)	0.135331	4.31194E-6

Example 3: Consider the equation (5) with $k(x,y,s,t) = xyst$ and $f(x,y) = \frac{1}{9}xy$, where the exact solution is $u(x,y) = xy$. Numerical results are shown in Table 2.

Example 4: For the two-dimensional Fredholm integral equation of the second kind (6) with the kernel function $k(x,y,s,t) = \frac{x}{1+y}(1+s+t)$, and $f(x,y) =$

$\frac{1}{1+x+y} - \frac{x}{1+y}$, the exact solution is $u(x,y) = \frac{1}{1+x+y}$. The numerical results are shown in Table 3.

Table 2: Results of Example 3

(x_i, y_i)	Approximation Values	Absolute Errors
(0.1,0.1)	0.016006699128	6.69128-5
(0.2,0.2)	0.0400677636	6.67636E-5
(0.3,0.3)	0.0900671139	6.71139E-5
(0.4,0.4)	0.16000672545	6.72545E-5
(0.5,0.5)	0.2500673985	2.9304E-5
(0.6,0.6)	0.3600675444	6.754447E-5
(0.7,0.7)	0.4900676914	6.76914E-5
(0.8,0.8)	0.6400678393	6.78393E-5
(0.9,0.9)	0.810067988	6.7988E-5
(1.0,1.0)	1.0000681374	6.81374E-5

Table 3: Results of Example 4

(x_i, y_i)	Approximation Values	Absolute Errors
(0.1,0.1)	0.833326	7.36439E-6
(0.2,0.2)	0.71481	4.4919E-6
(0.3,0.3)	0.624999	1.49527E-6
(0.4,0.4)	0.555554	1.36714
(0.5,0.5)	0.499999	9.5077E-7
(0.6,0.6)	0.454545	6.08746E-7
(0.7,0.7)	0.416666	4.92839E-7
(0.8,0.8)	0.384615	2.55225E-7
(0.9,0.9)	0.357142	5.51285E-7
(1.0,1.0)	0.333333	2.82497E-7

Example 5: In the third example, we consider the two-dimensional Fredholm integral equation (6) as follows.

$$u(x,y) = x\cos(y) - \frac{1}{6}\sin(1)(3+\sin(1)) + \int_0^1 \int_0^1 (s\sin(t)+1)u(s,t)dsdt \quad (55)$$

The exact solution is $u(x,y) = xcos(t)$. Table 4 shows the absolute values of error.

Example 6: Consider the two-dimensional Integral equation (6), such that

$$k(x,y,s,t) = \frac{x}{(8+y)(1+t+s)}, f(x,y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(8+y)}, 0 \leq x, y \leq 1, \tag{56}$$

and the exact solution is $u(x,y) = \frac{1}{(1+x+y)^2}$. Approximation solutions are shown in Table 5.

Table 4: Results of Example 5

(x_i, y_i)	Approximation Values	Absolute Errors
(0.1,0.1)	0.0995011	7.20115E-7
(0.2,0.2)	0.96014	3.58335E-7
(0.3,0.3)	0.286601	1.0544E-7
(0.4,0.4)	0.368426	1.50861E-6
(0.5,0.5)	0.438792	9.1098E-7
(0.6,0.6)	0.495202	1.07965E-6
(0.7,0.7)	0.53592	2.53637E-6
(0.8,0.8)	0.557367	1.96953E-6
(0.9,0.9)	0.559453	3.86979E-6
(1.0,1.0)	0.540306	4.01721E-6

Table 5: Results of Example 6

(x_i, y_i)	Approximation Values	Absolute Errors
(0.1,0.1)	0.694444	3.84664E-7
(0.2,0.2)	0.510195	9.23253E-6
(0.3,0.3)	0.39062	4.8171E-6
(0.4,0.4)	0.308638	3.6988E-6
(0.5,0.5)	0249998	1.57176E-6
(0.6,0.6)	0.206611	5.55463E-7
(0.7,0.7)	0.173611	5.41342E-7
(0.8,0.8)	0.147929	2.08553E-7
(0.9,0.9)	0.127551	4.2866E-7
(1.0,1.0)	0.111111	3.53384E-7

6 Concluding Remarks

We apply scaling function interpolation method to solve two-dimensional integral equations by using coiflets and provide convergence rates. The numerical results (Table 3 and Table 4) reported in the examples in section 5 show that our method has produced better results than those obtained by the method used in [11]. We also have better results than others as we compare our results shown in Table 5 with the results in [12]. The multiresolution and vanishing moments properties contribute to the robustness and accuracy of the methodology. It is promising to use our method to solve integral equations mentioned in section 4 for further image processing applications, which will be our future projects.

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References

[1] E. Lin and X. Zhou, Coiflet Interpolation and Approximation Solutions of Elliptic Partial Differential Equations, Number Methods Partial Differential Equations 13, 303-320, 1997

[2] K. Nikolay and B. Larisa, An Adaptive Finite Element Method for Fredholm Integral equations of the First Kind and its Verification on Experimental Data, Matematiska vetenskaper, Gteborg, 2012.

[3] E. Hernandez and G. Weiss, A First Course on Wavelets, CRC Press, 2010.

[4] C. S. Burrus, R. A. Gopinath and H. Guo, Introduction to Wavelets and Wavelet Transforms - A Primer. Prentice Hall, 1997.

[5] Y. Lu, L. Shen and Y. Xu, Integral equation methods for image restoration: high accuracy methods and fast algorithms. Inverse Problems 26(4), 1-30, 2010.

[6] C. Su and T. K. Sarker, Adaptive Multiscale Moment Method for Solving Two-dimensional Fredholm Integral equation of the First Kind. Progress In Electromagnetics research, PIER 21, 173-201, 1999.

[7] E. Lin and Y. Al-Jarrah, A Wavelet Based Method for the Solution of Fredholm Integral equations. American Journal of Computational Mathematics, 2(2012), 114-117.

[8] Y. Al-Jarrah and E. Lin, Numerical Solution of Fredholm-Volterra Integral Equations by Using Scaling Function Interpolation Method, Applied Mathematics, 4 (2013), 204-209.

[9] Y. Al-Jarrah and E. Lin, Numerical Solution of Fredholm Integral Equations of the First Kind by Using Coiflets, Journal of Modern Mathematics Frontier, 4 (2012), 1-6.

[10] H. Brunner, Houwen, P. J. van der Houwen, The Numerical Solution of Volterra Equation, Volume 3 of CWI Monographs, North-Holland Publishing Co., Amsterdam, 1986.

[11] H. Derili, S. Sohrabi and A. Arzhang, Two-Dimensional Wavelets for Numerical Solution of Integral Equations, Mathematical Sciences, a SpringerOpen Journal, 6:5, 1-4, 2012.

[12] A. Fallahzadeh, Solution of Two-Dimensional Fredholm Integral Equation via RBF-triangular Method. Journal of Interpolation and Approximation in Science Computing, Volume 2012, 1-5, 2012.

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