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Wave Function Solutions by Transformation from the Helmholtz to Laplacian Operator

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Abstract

The Helmholtz, Schrödinger and Klein-Gordon equations all have a similar form (for constant wavelength) and have applications in optics, quantum mechanics and relativistic quantum mechanics, respectively. Central to these applications is the theory of barrier and potential scattering, which, through application of the Green's function method yields transcendental equations for the scattered wave function thereby requiring approximation methods to be employed. This paper reports on a new approach to solving this problem which is based on transforming the Helmholtz operator to the Laplacian operator and applying a Green's function solution to the Poisson equation. This approach yields an exact forward and inverse scattering solution subject to a fundamental condition, whose physical basis is briefly explored. It also provides a series solution that is not predicated on a convergence condition.

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Helmholtz, Schrödinger, Klein-Gordon, Poisson, Scattering, Exact solutions.

1 Introduction

We consider three-dimensional solutions to the inhomogeneous equation

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = -f(\mathbf{r}, k)u(\mathbf{r}, k) \tag{1}$$

for the wave function $u(\mathbf{r}, k)$, where ∇^2 denotes the three-dimensional Laplacian operator, k is a constant (the wavenumber), $\mathbf{r}$ is the three-dimensional space vector and $f(\mathbf{r}, k)$ may be a piecewise continuous or a generalised function and may be real or complex. The precise form of $f(\mathbf{r}, k)$ in equation (1) determines the 'type' of (inhomogeneous) equation. This is detailed in Table 1 and includes applications in Electromagnetism and Acoustic (inhomogeneous Helmholtz equation), [1] and [2], Non-relativistic Quantum Mechanics (Schrödinger equation) [3] and Relativistic Quantum Mechanics (Klein-Gordon equation) [4], as per Table 1. In the latter case, the function $f(\mathbf{r}, k)$ is usually taken to represent an asymptotic potential whereas in applications such as electromagnetism the function is taken to be of compact support $\mathbf{r} \in \mathbb{R}^3$. In the context of the solutions to equation (1) considered in this paper, $f(\mathbf{r}, k)$ is a 'scattering function' of compact support or a scattering 'potential' of compact support or otherwise (an asymptotic potential) and the solution for $u(\mathbf{r}, k)$ determines the scattered wave function.

We consider conventional approaches to solving the scattering problem: *Given* $f(\mathbf{r}, k)$ *evaluate* $u(\mathbf{r}, k)$. This is done to contextualise the new approach that is considered. We then introduce a transformation which provides a complementary solution that is investigated further including the conditions required to provide an exact scattering solution. It is this transformation, and, the results thereof, which, to the best of the authors knowledge, constitutes an original contribution and is based on 're-casting' the LHS of equation (1), i.e. a transformation from the Helmholtz operator $\nabla^2 + k^2$ to the Laplacian operator ∇^2.

Table 1: Equations for different functions $f(\mathbf{r}, k)$ where $\epsilon_r(\mathbf{r})$ denotes the relative permittivity, λ is the wavelength, $\hbar$ is Planck's constant, c_0 is the speed of light in a vacuum, E and m denote the energy and mass of a particle, respectively, and $V(\mathbf{r})$ is the potential energy function.

Equation type	Function $f(\mathbf{r}.k)$	Wavenumber k	Conditions
Helmholtz	$k^2[\epsilon_r(\mathbf{r}) - 1]$	$\frac{2\pi}{\lambda}$	$\epsilon_r(\mathbf{r})\exists\forall\mathbf{r} \in \mathbb{R}^3$
Schrödinger	$-\frac{2mV(\mathbf{r})}{\hbar^2}$	$\pm\sqrt{\frac{2mE}{\hbar^2}}$	$V(\mathbf{r}) \to 0, r \to \infty$
Klein-Gordon	$-2E\left[V(\mathbf{r}) - \frac{V^2(\mathbf{r})}{2E}\right]$	$\pm\sqrt{E^2 - m^2c_0^4}$	$V(\mathbf{r}) \to 0, r \to \infty$

2 General Solution to Equation (1)

For $f(\mathbf{r}, k) \exists \forall \mathbf{r} \in \mathbb{R}^3$, the general solution to equation (1) is well known and given by [5]

$$u(\mathbf{r}, k) = S(\mathbf{r}, k) + g(r, k) \otimes_3 f(\mathbf{r}, k) u(\mathbf{r}, k) \tag{2}$$

where, for a unit vector $\hat{\mathbf{n}}$ that is perpendicular to the surface $\mathcal{S}$ defined by $\mathbb{R}^3$,

$$S(\mathbf{r}, k) = \oint_{\mathcal{S}} [g(\mathbf{r} \mid \mathbf{s}, k) \nabla u(\mathbf{s}, k) - u(\mathbf{s}, k) \nabla g(\mathbf{r} \mid \mathbf{s}, k)] \cdot \hat{\mathbf{n}} d^2\mathbf{s},$$

$g(r, k)$ is a 'out-going free space Green's function'

$$g(r, k) = \frac{\exp(ikr)}{4\pi r}$$

which is the solution of

$$(\nabla^2 + k^2) g(\mathbf{r} \mid \mathbf{s}, k) = -\delta^3(\mathbf{r} - \mathbf{s}), \quad \mathbf{r} \mid \mathbf{s} \ \equiv \mid \mathbf{r} - \mathbf{s} \mid$$

and $\otimes_3$ denotes the three-dimensional convolution integral

$$g(r, k) \otimes_3 f(\mathbf{r}, k) u(\mathbf{r}, k) \equiv \int_{\mathbb{R}^3} g(\mid \mathbf{r} - \mathbf{s} \mid, k) f(\mathbf{s}, k) u(\mathbf{s}, k) d^3\mathbf{s}$$

Remark 2.1

Although equation equation (2) is conventionally referred to as the Green's function 'solution' to equation (1) it is not strictly speaking a 'solution'. This is because equation (2) is a transcendental equation in $u(\mathbf{r}, k)$ and rather than being a 'solution', should be referred to as a Green's function 'transformation' from the partial differential equation given by equation (1) to the integral equation given by equation (2). This transformation is entirely general.

Remark 2.2

The surface integral given in equation (2), obtained through application of Green's Theorem in the Green's function solution to equation (1), characterises that part of the 'general transformation' to equation (1) that is due to the boundary effects generated by the surface $\mathcal{S}$ defined by $\mathbb{R}^3$. In this sense, the first term on the RHS of equation (2) describes the surface boundary effects generated by the function $f(\mathbf{r}, k)$ being of compact support.

2.1 Surface Integral Analysis

From equations (1) and (2), it is clear that if $f(\mathbf{r},k)=0\forall\mathbf{r}$. then

$$u(\mathbf{r},k)=\oint_{\mathcal{S}}[g(\mathbf{r}\mid\mathbf{s},k)\nabla u(\mathbf{s},k)-u(\mathbf{s},k)\nabla g(\mathbf{r}\mid\mathbf{s},k)]\cdot\hat{\mathbf{n}}d^2\mathbf{s} \tag{3}$$

which is the general solution to the homogeneous Helmholtz equation

$$(\nabla^2+k^2)u(\mathbf{r},k)=0 \tag{4}$$

However, if

$$u(\mathbf{r},k)=u_i(\mathbf{r},k)\forall\mathbf{r}\in\mathcal{S} \tag{5}$$

where $u_i(\mathbf{r},k)$ is a solution to equation (4), then, from equation (3),

$$u(\mathbf{r},k)=u_i(\mathbf{r},k)\forall\mathbf{r}\in\mathbb{R}^3$$

This result can be shown from Green's Theorem, since

$$\oint_{\mathcal{S}}[g(\mathbf{r}\mid\mathbf{s},k)\nabla u_i(\mathbf{s},k)-u_i(\mathbf{s},k)\nabla g(\mathbf{r}\mid\mathbf{s},k)]\cdot\hat{\mathbf{n}}d^2\mathbf{s}$$

$$=\int_{\mathbb{R}^3}[g(\mathbf{r}\mid\mathbf{s},k)\nabla^2u_i(\mathbf{s},k)-u_i(\mathbf{s},k)\nabla^2g(\mathbf{r}\mid\mathbf{s},k)]d^3\mathbf{s}$$

$$=\int_{\mathbb{R}^3}[-g(\mathbf{r}\mid\mathbf{s},k)k^2u_i(\mathbf{s},k)-u_i(\mathbf{s},k)[-\delta^3(\mathbf{r}-\mathbf{s})-k^2g(\mathbf{r}\mid\mathbf{s},k)]d^3\mathbf{s}$$

$$=\int_{\mathbb{R}^3}\delta^3(\mathbf{r}-\mathbf{s})u_i(\mathbf{s},k)d^3\mathbf{s}=u_i(\mathbf{r},k)$$

Thus, when $f(\mathbf{r},k)$ is of compact support, and, in addition, the boundary condition given by equation (5) holds true, the general solution to equation (1) is, from equation (2), given by

$$u(\mathbf{r},k)=u_i(\mathbf{r},k)+g(r,k)\otimes_3 f(\mathbf{r},k)u(\mathbf{r},k) \tag{6}$$

On the other hand, if $f(\mathbf{r},k)$ is a potential of no compact support with asymptotic conditions as stated in Table 1, then the surface integral vanishes (the surface being taken to be taken at infinity) and the solution to equation (1) is also given by equation (6), the wave function u_i being the solution to equation (1) when the potential energy is zero and equation (1) reduces to equation (4). Thus we consider equation (6) to be the general solution to equation (1) for all cases given in Table 1 subject to the boundary condition (5) when $f(\mathbf{r},k)$ is of compact support and where the three-dimensional convolution integral, denoted by $\otimes_3$, can be over either finite space (functions of compact support) or over all space (asymptotic potentials).

2.2 Analysis of Equation (6)

If we apply the Helmholtz operator $\nabla^2 + k^2$ to equation (6) it is clear that

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = (\nabla^2 + k^2)u_i(\mathbf{r}, k) + (\nabla^2 + k^2)[g(r, k) \otimes_3 f(\mathbf{r}, k)u(\mathbf{r}, k)]$$

$$= [(\nabla^2+k^2)g(r, k)]\otimes_3 f(\mathbf{r}, k)u(\mathbf{r}, k) = -\delta^3(\mathbf{r})\otimes_3 f(\mathbf{r}, k)u(\mathbf{r}, k) = -f(\mathbf{r}, k)u(\mathbf{r}, k)$$

using the sampling property of the delta function and given that

$$(\nabla^2 + k^2)u_i(\mathbf{r}, k) = 0$$

Given this result, if we write the wave function in terms of the sum of $u_i(\mathbf{r}, k)$ and $u_s(\mathbf{r}, k)$, i.e.

$$u(\mathbf{r}, k) = u_i(\mathbf{r}, k) + u_s(\mathbf{r}, k)$$

then from equation (6), it is clear that we can write

$$u_s(\mathbf{r}, k) = g(r, k) \otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k) + g(r, k) \otimes_3 f(\mathbf{r}, k)u_s(\mathbf{r}, k) \tag{7}$$

The wave function $u_s(\mathbf{r}, k)$ is generally known as the scattered wave field and is clearly determined by the scattering function $f(\mathbf{r}, k)$.

Equation (7) is transcendental with regard to this field. As with other transcendental equations, solutions to equation (7) are predicated on the application of approximation methods or iterative techniques and other numerical methods. Central to this, is application of the approximation

$$u_s(\mathbf{r}, k) = g(r, k) \otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k) \tag{8}$$

which is known as the Born approximation and is the first iterate of the Born series which can be written in the form

$$u_s(\mathbf{r}, k) = g(r, k)\otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k)+g(r, k)\otimes_3 f(\mathbf{r}, k)[g(r, k)\otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k)]+... \tag{9}$$

The Born approximation has applications in a wide range of engineering application including optics, acoustics and particle physics. Physically, this approximation means that multiple scattering effects, modelled by the higher order terms associated with the series given in equation (9), are insignificant in the determination of $u_s(\mathbf{r}, k)$. This requires that

$$\frac{\|u_s(\mathbf{r}, k)\|}{\|u_i(\mathbf{r}, k)\|} << 1$$

which is not always strictly applicable. The inversion of equation (8), i.e. solving the inverse scattering problem: *Given* $u_s(\mathbf{r}, k)$ *solve for* $f(\mathbf{r}, k)$, whatever the geometric conditions applied, is therefore predicated on an approximation.

Other issues concerned with equation (7) include investigating 'Null Scattering Conditions' where certain scattering functions $f(\mathbf{r},k) \neq 0 \forall(\mathbf{r},k)$ yield a solution of the from $u_s(\mathbf{r},k)=0$.

Now that the conventional solution method has been defined, in the following section, we introduce a complementary solution that follows the same basic approach but provides a new solution to the problem which makes it possible to exercise a novel condition for which an exact scattering and thereby an exact inverse scattering solution is available.

3 Transformation to the Poisson Equation

From equation (7), it is clear that upon application of the Helmholtz operator ∇^2+k^2

$$(\nabla^2+k^2)u_s(\mathbf{r},k) = -f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)] \tag{10}$$

which yields equation (1) given that $(\nabla^2+k^2)u_i(\mathbf{r},k)=0$ and $u(\mathbf{r},k)=u_i(\mathbf{r},k)+u_s(\mathbf{r},k)$. We now note that

$$(\nabla^2+k^2)u_s(\mathbf{r},k) = \nabla^2\left[u_s(\mathbf{r},k) - \frac{k^2}{4\pi r}\otimes_3 u_s(\mathbf{r},k)\right] \tag{11}$$

since

$$\nabla^2\frac{1}{4\pi r} = -\delta^3(\mathbf{r})$$

Equation (10) is the principal identity upon which we now build a complementary solution to equation (1) as given by equation (7). Combining equations (10) and (11) we can write

$$\nabla^2\left[u_s(\mathbf{r},k) - \frac{k^2}{4\pi r}\otimes_3 u_s(\mathbf{r},k)\right] = -f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)]$$

which is Poisson's equations with Green's function solution

$$u_s(\mathbf{r},k) - \frac{k^2}{4\pi r}\otimes_3 u_s(\mathbf{r},k) = \frac{1}{4\pi r}\otimes_3 f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)] \tag{12}$$

since, taking the Laplacian of the RHS of equation (12),

$$\nabla^2\left(\frac{1}{4\pi r}\otimes_3 f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)]\right) = -\delta^3(\mathbf{r})\otimes_3 f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)]$$

$$= f(\mathbf{r},k)[u_i(\mathbf{r},k)+u_s(\mathbf{r},k)]$$

We can now rearrange equation (12) to obtain

$$u_s(\mathbf{r},k) = \frac{1}{4\pi r}\otimes_3 [f(\mathbf{r},k)u_i(\mathbf{r},k)] + \frac{1}{4\pi r}\otimes_3 [k^2+f(\mathbf{r},k)]u_s(\mathbf{r},k) \tag{13}$$

Like equation (7), equation (13) is transcendental with regard to the wave function $u_s(\mathbf{r}, k)$ and can be solved iteratively giving the solution

$$u_s(\mathbf{r}, k) = \frac{1}{4\pi r} \otimes_3 [f(\mathbf{r}, k)u_i(\mathbf{r}, k)] + \frac{1}{4\pi r} \otimes_3 [k^2 + f(\mathbf{r}, k)]u_i(\mathbf{r}, k)$$
$$+\frac{1}{4\pi r} \otimes_3 [k^2 + f(\mathbf{r}, k)] \left[\frac{1}{4\pi r} \otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k)\right] +$$
$$+\frac{1}{4\pi r} \otimes_3 [k^2 + f(\mathbf{r}, k)] \left[\frac{1}{4\pi r} \otimes_3 [k^2 + f(\mathbf{r}, k)]u_i(\mathbf{r}, k)\right] + ...$$

4 An Exact Scattering Solution

Both equations (7) and (13) are exact transformations of equation (1) into integral equations given that $u(\mathbf{r}, k) = u_i(\mathbf{r}, k) + u_s(\mathbf{r}, k)$ where $u_i(\mathbf{r}, k)$ is a solution to the homogeneous Helmholtz equation. Both equations are transcendental in $u_s(\mathbf{r}, k)$ and as such do not possess an exact solution. However, unlike equation (7), equation (13) provides us with a non-conventional condition under which its transcendental characteristics are eliminated. Through inspection of equation (13), it is clear that if

$$k^2 + f(\mathbf{r}, k) = 0 \tag{14}$$

then

$$u_s(\mathbf{r}, k) = \frac{1}{4\pi r} \otimes_3 [f(\mathbf{r}, k)u_i(\mathbf{r}, k)] \tag{15}$$

which is an exact solution to the problem. We therefore now consider the ramifications of equation (14) within the context of Table 1.

4.1 Helmholtz Scattering Problems

Condition (14) reduces to

$$\epsilon_r(\mathbf{r}) = 0 \forall \mathbf{r} \in \mathbb{R}^3$$

and equation (15) becomes

$$u_s(\mathbf{r}, k) = -\frac{k^2}{4\pi r} \otimes_3 u_i(\mathbf{r}, k) \tag{15}$$

Form a physical point of view, this is the solution for the scattering of a scalar electromagnetic field from a zero permittivity material.

Zero permittivity materials have been considered from a theoretical view point and create band gaps in a wide range of frequencies up to the visible. Their physical realisation is achieved through generating composite materials by embedding metallic nano-particles and nano-wires in a dielectric medium [6].

4.2 Schrödinger Scattering Problems

Condition (14) reduces to

$$V(\mathbf{r}) = E$$

and

$$u_s(\mathbf{r}, k) = \frac{1}{4\pi r} \otimes_3 u_i(\mathbf{r}, k) \tag{16}$$

The potential energy is therefore a constant, equal to the energy of a non-relativistic particle which may be over the region of space $\mathbb{R}^3$, i.e. a uniform potential of compact support.

4.3 Klein-Gordon Scattering Problems

Condition (14) reduces to

$$V(\mathbf{r}) = E \pm mc_0^2$$

the scattered field being given by equation (16). The potential energy is therefore a constant equal to the energy of a relativistic particle which may be over the region of space $\mathbb{R}^3$ (a uniform potential of compact support).

Remark 4.1

Although the conditions required to obtain equations (15) and (16) are highly specific, there is no condition on the topology defined by $\mathbb{R}^3$. Thus, for example, in the case of equation (15) a zero permittivity material can be of any shape and topological complexity. This is also the case if the potentials associated with the Schrödinger and Klein-Gordon scattering problems are of finite support.

Remark 4.2

If the conditions required for equation (15) and (16) are met, then the inversion of these equations represents and exact inverse scattering solution which will depend on the characteristics of the function $u_i(\mathbf{r}, k)$. If, for example, $u_i(\mathbf{r}, k) = \exp(ik\hat{\mathbf{n}} \cdot \mathbf{r})$ where $\hat{\mathbf{n}}$ is the unit vector that points in the direction of a plane wave that is incident on the scatterer, then, from equation (15)

$$u_s(\mathbf{r}, k) = -\frac{k^2}{4\pi} \int_{-\infty}^{\infty} O(\mathbf{r}) \frac{\exp(ik\hat{\mathbf{n}} \cdot \mathbf{s})}{\mid \mathbf{r} - \mathbf{s} \mid} d^3\mathbf{s} = -\frac{k^2}{4\pi r} \int_{-\infty}^{\infty} O(\mathbf{r}) \exp(ik\hat{\mathbf{n}} \cdot \mathbf{s}) d^3\mathbf{s}, \quad r \to \infty$$

where $O(\mathbf{r})$ is the unit 'Object Function'

$$O(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \mathbb{R}^3; \\ 0, & \mathbf{r} \notin \mathbb{R}^3. \end{cases}$$

composed of a material whose relative permittivity (at wavenumber k) is zero. In this case, the exact inverse scattering solution is compounded in the inverse Fourier transform. Further, since this is an exact solution, under the condition of the Object Function being a zero permittivity material, such a material will represent a 'perfect reflector' of electromagnetic waves as confirmed by [6] based on transmission calculations using a finite-difference solution. Such an effect is not realised using equation (7).

5 On the Simultaneity of Equations (7) and (13)

Equations (7) and (13) are simultaneous integral equations for $u_s(\mathbf{r}, k)$.

Theorem 5.1

The simultaneity of equations (7) and (13) is consistent with equation (1) given that $u(\mathbf{r}, k) = u_i(\mathbf{r}, k) + u_s(\mathbf{r}, k)$ and

$$(\nabla^2 + k^2)u_i(\mathbf{r}, k) = 0$$

Proof

Subtracting equation (13) from equation (7) is clear that

$$g(r, k) \otimes_3 f(\mathbf{r}, k)u_i(\mathbf{r}, k) + g(r, k) \otimes_3 f(\mathbf{r}, k)u_s(\mathbf{r}, k)$$

$$-\frac{1}{4\pi r} \otimes_3 [f(\mathbf{r}, k)u_i(\mathbf{r}, k)] - \frac{1}{4\pi r} \otimes_3 [k^2 + f(\mathbf{r}, k)]u_s(\mathbf{r}, k) = 0$$

Taking the three-dimensional Fourier transform of this equation and using the convolution and product theorems we have

$$\frac{1}{p^2 - k^2}[\widetilde{f}(\mathbf{p}, k) \otimes_3 \widetilde{u}_i(\mathbf{p}, k)] + \frac{1}{p^2 - k^2}[\widetilde{f}(\mathbf{p}, k) \otimes_3 \widetilde{u}_s(\mathbf{p}, k)]$$

$$-\frac{1}{p^2}[\widetilde{f}(\mathbf{p}, k) \otimes_3 \widetilde{u}_i(\mathbf{p}, k)] - \frac{k^2}{p^2}\widetilde{u}_s(\mathbf{p}, k) - \frac{1}{p^2}[\widetilde{f}(\mathbf{p}, k) \otimes_3 \widetilde{u}_s(\mathbf{p}, k)] = 0$$

where

$$\widetilde{u}_i(\mathbf{p}, k) = \int_{-\infty}^{\infty} u_i(\mathbf{r}, k) \exp(-i\mathbf{p} \cdot \mathbf{r}) d^3\mathbf{r}, \quad \widetilde{u}_s(\mathbf{p}, k) = \int_{-\infty}^{\infty} u_s(\mathbf{r}, k) \exp(-i\mathbf{p} \cdot \mathbf{r}) d^3\mathbf{r},$$

$$\widetilde{f}(\mathbf{p}, k) = \int_{-\infty}^{\infty} f(\mathbf{r}, k) \exp(-i\mathbf{p} \cdot \mathbf{r}) d^3\mathbf{r}$$

and $\otimes_3$ is now taken to denote the three-dimensional convolution integral over $\mathbf{p}$. Rearranging,

$$-p^2\widetilde{u}_s(\mathbf{p},k)+k^2\widetilde{u}_s(\mathbf{p},k)=-\widetilde{f}(\mathbf{p},k)\otimes_3[\widetilde{u}_i(\mathbf{p},k)+\widetilde{u}_s(\mathbf{p},k)]$$

and upon inverse Fourier transformation

$$(\nabla^2+k^2)u_s(\mathbf{r},k)=-f(\mathbf{r},k)u(\mathbf{r},k)$$

Given that equations (7) and (13) are consistent with equation (1) we can exploit their simultaneity do develop a series solution. Both equation (7) and equation (13) can be solved iteratively but this requires that the convergence condition

$$\frac{\|u_s(\mathbf{r},k)\|}{\|u_i(\mathbf{r},k)\|}<1$$

is valid. However, given that equations (7) and (13) can be taken to be solutions to the same equation without loss of generality, a series solution can be developed that is independent of this condition. For example, we can consider the following result

$$u_s(\mathbf{r},k)\simeq g(r,k)\otimes_3 f(\mathbf{r},k)u_i(\mathbf{r},k)+g(r,k)\otimes_3 f(\mathbf{r},k)\left(\frac{1}{4\pi r}\otimes_3[f(\mathbf{r},k)u_i(\mathbf{r},k)]\right)$$

which is based on approximating equation (13) by ignoring the second term on the RHS and substituting the result into equation (7). Similarly, we can write

$$u_s(\mathbf{r},k)\simeq\frac{1}{4\pi r}\otimes_3[f(\mathbf{r},k)u_i(\mathbf{r},k)]+\frac{1}{4\pi r}\otimes_3[g(r,k)\otimes_3 f(\mathbf{r},k)u_i(\mathbf{r},k)]$$

obtained by applying the Born approximation to equation (7) by ignoring the second term on the RHS and substitute the result into equation (13). These results provide expressions for the scattered wave function that transcends existing solutions under the Born approximation.

6 Exact Series Solutions

We can repeat the approach used to obtain the approximate solutions for $u_s(\mathbf{r},k)$ given in the last section. This allows us to develop the following convergence condition independent series solution for equation (7) using equation (13) by interchanging the expression for $u_s(\mathbf{r},k)$ from each equation on a consecutive basis:

$$u_s(\mathbf{r},k)=g(r,k)\otimes_3 f(\mathbf{r},k)u_i(\mathbf{r},k)+g(r,k)\otimes_3 f(\mathbf{r},k)\left(\frac{1}{4\pi r}\otimes_3[f(\mathbf{r},k)u_i(\mathbf{r},k)]\right)$$

$$+g(r,k)\otimes_3 f(\mathbf{r},k)\left(\frac{1}{4\pi r}\otimes_3[k^2+f(\mathbf{r},k)]u_s(\mathbf{r},k)\right)$$

$$+g(r,k)\otimes_3 f(\mathbf{r},k)\left[\frac{1}{4\pi r}\otimes_3[k^2+f(\mathbf{r},k)]\left(\frac{1}{4\pi r}\otimes_3 f(\mathbf{r},k)u_i(\mathbf{r},k)\right)\right]+...$$

However, the same principle can be used to develop an inverse solution to the problem as follows. Consider a function $\phi(\mathbf{r},k)$ for which

$$\phi(\mathbf{r},k)\otimes_3[g(r,k)\otimes_3 f(\mathbf{r},k)u_i(\mathbf{r},k)]=f(\mathbf{r},k)$$

and a function $\psi(\mathbf{r},k)$ such that

$$\psi(\mathbf{r},k)\otimes_3\left(\frac{1}{4\pi r}\otimes_3[f(\mathbf{r},k)u_i(\mathbf{r},k)]\right)=f(\mathbf{r},k)$$

We can then write equations (7) and (13) in the form

$$f(\mathbf{r},k)=\phi(\mathbf{r},k)\otimes_3 u_s(\mathbf{r},k)-\phi(\mathbf{r},k)\otimes_3[g(r,k)\otimes_3 f(\mathbf{r},k)u_s(\mathbf{r},k)]$$

and

$$f(\mathbf{r},k)=\psi(\mathbf{r},k)\otimes_3 u_s(\mathbf{r},k)-\psi(\mathbf{r},k)\otimes_3\left(\frac{1}{4\pi r}\otimes_3[k^2+f(\mathbf{r},k)]u_s(\mathbf{r},k)\right)$$

respectively. Interchanging the expression for $f(\mathbf{r},k)$ on a consecutive basis, we can then write (up to three terms in the series)

$$f(\mathbf{r},k)=\phi(\mathbf{r},k)\otimes_3 u_s(\mathbf{r},k)-\phi(\mathbf{r},k)\otimes_3[g(r,k)\otimes_3[\psi(\mathbf{r},k)\otimes_3 u_s(\mathbf{r},k)]u_s(\mathbf{r},k)]$$

$$+\phi(\mathbf{r},k)\otimes_3[g(r,k)\otimes_3\psi(\mathbf{r},k)\otimes_3\left(\frac{1}{4\pi r}\otimes_3[k^2+\phi(\mathbf{r},k)\otimes_3 u_s(\mathbf{r},k)]u_s(\mathbf{r},k)\right)-...$$

7 Conclusion

The kernel of the new result reported in this paper is compounded in the following Theorem.

Theorem 6.1

Given that equation (6) is a solution equation (1) without loss of generality, equation (1) can be written in the form

$$\nabla^2\left[\frac{k^2}{4\pi r}\otimes_3 u_s(\mathbf{r},k)+u_s(\mathbf{r},k)\right]=-f(\mathbf{r},k)u(\mathbf{r},k)$$

without loss of generality.

Proof

From equation (6), we can write

$$u_s(\mathbf{r},k) = g(r.k) \otimes_3 f(\mathbf{r},k)u(\mathbf{r},k)$$

where

$$u_s(\mathbf{r},k) = u(\mathbf{r},k) - u_i(\mathbf{r},k)$$

Let $q(\mathbf{r},k)$ be an auxiliary function such that

$$q(\mathbf{r},k) \otimes_3 u_s(\mathbf{r},k) = q(\mathbf{r}.k) \otimes_3 [g(r,k) \otimes_3 f(\mathbf{r},k)u(\mathbf{r},k)]$$

Taking the Laplacian operator of this equation,

$$\nabla^2[q(\mathbf{r},k) \otimes_3 u_s(\mathbf{r},k)] = \nabla^2[q(\mathbf{r},k) \otimes_3 g(r.k) \otimes_3 f(\mathbf{r},k)u(\mathbf{r},k)]$$
$$= \nabla^2[q(\mathbf{r},k) \otimes_3 g(r.k)] \otimes_3 f(\mathbf{r},k)u(\mathbf{r},k) = -f(\mathbf{r},k)u(\mathbf{r},k)$$

provided

$$\nabla^2[q(\mathbf{r},k) \otimes_3 g(r,k)] = -\delta^3(\mathbf{r})$$

Now

$$\nabla^2[q(\mathbf{r},k) \otimes_3 g(r,k)] = q(\mathbf{r},k) \otimes_3 \nabla^2 g(r,k)$$
$$= q(\mathbf{r},k) \otimes_3 [-k^2 g(r,k) - \delta^3(\mathbf{r})] = -k^2 q(\mathbf{r},k) \otimes_3 g(r,k) - q(\mathbf{r},k)$$

and hence

$$q(\mathbf{r},k) = \delta^3(\mathbf{r}) - k^2 q(\mathbf{r},k) \otimes_3 g(r,k)$$

so that

$$\nabla^2[q(\mathbf{r},k) \otimes_3 u_s(\mathbf{r},k)]$$
$$= \nabla^2[\delta^3(\mathbf{r}) \otimes_3 u_s(\mathbf{r},k) - k^2 q(\mathbf{r},k) \otimes_3 g(r,k) \otimes_3 u_s(\mathbf{r},k)]$$

$$= \nabla^2[u_s(\mathbf{r},k) - k^2 q(\mathbf{r},k) \otimes_3 g(r,k) \otimes_3 u_s(\mathbf{r},k)] = -f(\mathbf{r},k)u(\mathbf{r},k)$$

where $q(\mathbf{r},k)$ is determined by the solution of

$$\nabla^2[q(\mathbf{r},k) \otimes_3 g(r,k)] = -\delta^3(\mathbf{r})$$

which is given by

$$q(\mathbf{r},k) \otimes_3 g(r,k) = \frac{1}{4\pi r} \tag{17}$$

so that

$$\nabla^2 \left[u_s(\mathbf{r}, k) - \frac{k^2}{4\pi r} \otimes_3 u_s(\mathbf{r}, k) \right] = -f(\mathbf{r}, k) u(\mathbf{r}, k)$$

This completes the proof of Theorem 6.1 subject to a further proof that the auxiliary function $q(\mathbf{r}, k)$ exists, i.e. that there exists a solution to equation (17). A general solution for $q(\mathbf{r}, k)$ is therefore provided in the following Theorem.

Theorem 6.2

Given equation (17), the solution for $q(\mathbf{r}, k)$ is

$$q(\mathbf{r}, k) = \delta^3(\mathbf{r}) - \frac{k^2}{4\pi r}$$

Proof

Taking the Laplacian of equation (17), we have

$$g(r, k) \otimes_3 \nabla^2 q(\mathbf{r}, k) = -\delta^3(\mathbf{r}) \tag{18}$$

Taking the Fourier transform of equation (18), we obtain

$$\frac{p^2}{p^2 - k^2} Q(\mathbf{p}) = 1, \quad p \neq k$$

where

$$Q(\mathbf{p}) = \int_{-\infty}^{\infty} q(\mathbf{r}, k) \exp(-i\mathbf{p} \cdot \mathbf{r}) d^3\mathbf{r}$$

Thus, using spherical polar coordinates (r, θ, ϕ),

$$\begin{aligned}
q(\mathbf{r},k) &= \frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\left(1-\frac{k^2}{p^2}\right)\exp(i\mathbf{p}\cdot\mathbf{r})d^3\mathbf{p} \\
&= \delta^3(\mathbf{r}) - \frac{k^2}{(2\pi)^3}\int\limits_0^{2\pi} d\phi \int\limits_{-1}^{1} d(\cos\theta)\int\limits_0^{\infty} du\exp(ipr\cos\theta) \\
&= \delta^3(\mathbf{r}) - \frac{k^2}{2\pi^2 r}\int\limits_0^{\infty}\frac{\sin(pr)}{p}dp \\
&= \delta^3(\mathbf{r}) - \frac{k^2}{4\pi r} \qquad \left(\int\limits_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}\right)
\end{aligned}$$

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