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Various operators and fuzzy relation in complete residuated lattices

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Abstract

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators in a complete residuated lattice. In particular, we study the relationships between fuzzy relations and various operators.

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1 Introduction

Wille [9] introduced the structures on lattices by allowing some uncertainty in data. The structures on lattices are important mathematical tools for data analysis and knowledge processing [1,2,5-7,9]. Kim [5,6] investigated the properties of modal, necessity, sufficiency and co-sufficiency operators on sets. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [1,4,8].

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators in a complete residuated lattice. In particular, we study the relationships between fuzzy relations and various operators.

2 Preliminaries

Definition 2.1 [3] A triple $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) It has an adjointness ,i.e.

$$x \leq y \to z \text{ iff } x \odot y \leq z.$$

A map $^* : L \to L$ defined by $a^* = a \to 0$ is called a *strong negation* if $b^* \leq a^*$ for $a \leq b$ and $a^{**} = a$.

Example 2.2 [1,3,4,8] (1) Each frame $(L, \vee, \wedge, \odot = \wedge, \to, 0, 1)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm $\odot$, $([0,1], \vee, \wedge, \odot, \to, 0, 1)$, is a complete residuated lattice.

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, 0, 1)$ be a complete residuated lattice with strong negation *.

Lemma 2.3 [1,3,4,8] *For each $x, y, z, x_i, y_i \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then the following properties hold.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(2) *$x \odot y \leq x \wedge y$ and $x \odot (x \to y) \leq y$.*

(3) *$x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.*

(4) *$x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.*

(5) *$(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.*

(6) *$x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$*

(7) *$(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.*

(8) *$\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.*

(9) *$(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.*

(10) *$1 \to x = x$ and $x \to y = y^* \to x^*$.*

(11) *$x \leq y$ iff $x \to y = 1$.*

(12) *$(x \to y) \odot (y \to z) \leq x \to z$.*

(13) *$(x \to y)^* = x \odot y^*$.*

3 Various operators and fuzzy relation in complete residuated lattices

Definition 3.1 *A map $F : L^X \to L^Y$ is called:*

(1) *modal operator if $F(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i)$, $F(\alpha \odot A) = \alpha \odot F(A)$,*

(2) *necessity operator if $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$, $F(\alpha \to A) = \alpha \to F(A)$,*

(3) *sufficiency operator if* $F(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i),\ F(\alpha \odot A) = \alpha \to F(A)$,

(4) *co-sufficiency operator if* $F(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i), F(\alpha \to A) = \alpha \odot F(A)$.

(5) *If* $F : L^X \to L^X$ *is a map, then its dual operator* F^∂ *is defined by* $F^\partial(A) = F(A^*)^*$ *where* $A^*(x) = A(x) \to 0$*. Moreover, its complementary counterpart* $F^*(A) = (F(A))^*$ *and* $F^c(A) = F(A^*)$.

Remark 3.2 [2,5,6,7] *Let* $L = \{0, 1\}$ *be given. We regard* L^X, L^Y *as* $P(X), P(Y)$, *respectively. Then a map* $F : P(X) \to P(Y)$ *is called*

(1) *a modal operator if* $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(\emptyset) = \emptyset$.

(2) *a necessity operator if* $F(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(X) = Y$.

(3) *a sufficiency operator if* $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(\emptyset) = Y$.

(4) *a co-sufficiency operator if* $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(X) = \emptyset$.

(5) *a dual operator* F^∂ *is defined by* $F^\partial(A) = F(A^c)^c$*. Moreover, its complementary counterpart* $F^*(A) = (F(A))^c$ *and* $F^c(A) = F(A^c)$.

Definition 3.3 Let $R \in L^{X\times Y}$ be a fuzzy relation. For each $A \in L^X$, we define operations $R^{-1}(y, x) = R(x, y)$ and $[R], [[R]], \langle R\rangle, \langle\langle R\rangle\rangle, [R]^c, \langle R\rangle^c : L^X \to L^Y$ as follows:

$$[R](A)(y) = \bigwedge_{x\in X} (R(x, y) \to A(x)),\quad [[R]](A)(y) = \bigwedge_{x\in X} (A(x) \to R(x, y)),$$

$$\langle R\rangle(A)(y) = \bigvee_{x\in X} (R(x, y) \odot A(x)),\quad \langle\langle R\rangle\rangle(A)(y) = \bigvee_{x\in X} (R^*(x, y) \odot A^*(x)),$$

$$[R]^c(A)(y) = \bigwedge_{x\in X} (R(x, y) \to A^*(x)),\quad \langle R\rangle^c(A)(y) = \bigvee_{x\in X} (R(x, y) \odot A^*(x)).$$

Theorem 3.4 (1) *A map* $F : L^X \to L^X$ *is a modal operator iff* $F^\partial : L^X \to L^X$ *is a necessity operator.*

(2) *A map* $F : L^X \to L^X$ *is a sufficiency operator iff* $F^\partial : L^X \to L^X$ *is a co-sufficiency operator operator.*

(3) *A map* $F : L^X \to L^X$ *is a modal operator iff* $F^c : L^X \to L^X$ *is a sufficient operator.*

(4) *A map* $F : L^X \to L^X$ *is a sufficiency operator iff* $F^c : L^X \to L^X$ *is a necessity operator operator.*

(5) *A map* $F : L^X \to L^X$ *is a modal operator iff* $F^* : L^X \to L^X$ *is a cosufficient operator.*

(6) *A map* $F : L^X \to L^X$ *is a sufficiency operator iff* $F^* : L^X \to L^X$ *is a necessity operator operator.*

Proof. (1) Let $F : L^X \to L^X$ be a modal operator.

$$\begin{array}{ll} F^\partial(\bigwedge_{i\in\Gamma} A_i) & = \Big(F(\bigvee_{i\in\Gamma} A_i^*)\Big)^* = \Big(\bigvee_{i\in\Gamma} F(A_i^*)\Big)^* \\ & = \bigwedge_{i\in\Gamma}(F(A_i^*))^* = \bigwedge_{i\in\Gamma} F^\partial(A_i) \end{array}$$

$$\begin{array}{ll} F^\partial(\alpha \to A) & = \Big(F((\alpha\to A)^*)\Big)^* = \Big(F(\alpha\odot A^*)\Big)^* \text{ (by Lemma 2.3 (13))} \\ & = \Big(\alpha\odot F(A^*)\Big)^* = \alpha\to F(A^*)^* = \alpha\to F^\partial(A). \end{array}$$

Conversely, $(F^\partial)^\partial(A) = (F^\partial(A^*))^* = F(A)$.

$$\begin{array}{ll} F(\bigvee_{i\in\Gamma} A_i) & = \Big(F^\partial(\bigwedge_{i\in\Gamma} A_i^*)\Big)^* = \Big(\bigwedge_{i\in\Gamma} F^\partial(A_i^*)\Big)^* \\ & = \bigvee_{i\in\Gamma}(F^\partial(A_i^*))^* = \bigvee_{i\in\Gamma} F(A_i) \end{array}$$

$$\begin{array}{ll} F(\alpha\odot A) & = \Big(F^\partial((\alpha\odot A)^*)\Big)^* = \Big(F^\partial(\alpha\to A^*)\Big)^* \\ & = \Big(\alpha\to F^\partial(A^*)\Big)^* = \alpha\odot F^\partial(A^*)^* = \alpha\odot F(A). \end{array}$$

(4) Let $F : L^X \to L^X$ be a sufficiency operator.

$$\begin{array}{ll} F^c(\bigwedge_{i\in\Gamma} A_i) & = F((\bigwedge_{i\in\Gamma} A_i)^*) == F(\bigvee_{i\in\Gamma} A_i^*) \\ & = \bigwedge_{i\in\Gamma} F(A_i^*) = \bigwedge_{i\in\Gamma} F^c(A_i) \end{array}$$

$$\begin{array}{ll} F^c(\alpha\to A) & = F((\alpha\to A)^*) = F(\alpha\odot A^*) \\ & = \alpha\to F(A^*) = \alpha\to F^c(A). \end{array}$$

Conversely,

$$\begin{array}{ll} F(\bigvee_{i\in\Gamma} A_i) & = F^c(\bigwedge_{i\in\Gamma} A_i^*) = \bigwedge_{i\in\Gamma} F^c(A_i^*) \\ & = \bigwedge_{i\in\Gamma} F(A_i) \end{array}$$

$$\begin{array}{ll} F(\alpha\odot A) & = F^c((\alpha\odot A)^*) = F^c(\alpha\to A^*) \\ & = \alpha\to F^c(A^*) = \alpha\to F(A). \end{array}$$

Other cases are similarly proved.

Theorem 3.5 *Let $F, G : L^X \to L^Y$ be operators. Then the following properties hold:*

(1) $(F^\partial)^\partial = F$, $(F^*)^* = F$ *and* $(F^c)^c = F$.

(2) $(F^\partial)^* = (F^*)^\partial$, $(F^\partial)^c = (F^c)^\partial$ *and* $(F^*)^c = (F^c)^* = F^\partial$.

(3) $(F\vee G)^\partial = F^\partial\wedge G^\partial$, $(F\vee G)^* = F^*\wedge G^*$ *and* $(F\vee G)^c = F^c\vee G^c$.

(4) *$F, G : L^X \to L^Y$ are modal operators, then $F\vee G$ is a model operator and it's dual operator $F^\partial\wedge G^\partial$ is a necessity operator.*

(5) *$F, G : L^X \to L^Y$ are necessity operators, then $F\wedge G$ is a necessity operator and it's dual operator $F^\partial\vee G^\partial$ is a model operator.*

Proof. (1) $(F^\partial)^\partial(A) = (F^\partial(A^*))^* = F(A)$.

(2) $(F^\partial)^*(A) = (F^\partial(A))^* = F(A^*) = F^c(A) = (F^*(A^*))^* = (F^*)^\partial(A)$.

(3) $(F\vee G)^\partial(A) = ((F\vee G)(A^*))^* = (F(A^*))^* \wedge (F(A^*))^* = F^\partial(A)\wedge G^\partial(A)$.

(4) Since F, G are modal operators, then $(F \vee G)(\bigvee_{i\in\Gamma} A_i) = F(\bigvee_{i\in\Gamma} A_i) \vee G(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i) \vee \bigvee_{i\in\Gamma} G(A_i) = \bigvee_{i\in\Gamma}(F\vee G)(A_i)$ and $(F\vee G)(\alpha\odot A) = \alpha \odot (F \vee G)(A)$. Hence $F \vee G$ is a model operator.

(5) Since F, G are necessity operators, then $(F\wedge G)(\bigwedge_{i\in\Gamma} A_i) = F(\bigwedge_{i\in\Gamma} A_i) \wedge G(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i) \wedge \bigwedge_{i\in\Gamma} G(A_i) = \bigwedge_{i\in\Gamma}(F \wedge G)(A_i)$ and $(F \wedge G)(\alpha \to A) = F(\alpha \to A) \wedge G(\alpha \to A) = (\alpha \to F(A)) \wedge (\alpha \to G(A)) = \alpha \to (F \wedge G)(A)$. Hence $F \wedge G$ is a necessity operator.

Other cases are similarly proved.

We denote a function 1_x defined by $1_x(x) = 1$ and $1_x(y) = 0$ for $x \neq y$.

Theorem 3.6 *Let $R \in L^{X\times Y}$ be a fuzzy relation.*

(1) $\langle R\rangle$ is a modal operator and $[R]$ is a necessity operator with $\langle R\rangle(A) = ([R](A^*))^* = [R]^\partial(A)$, for each $A \in L^X$

(2) If $F : L^X \to L^Y$ is a modal operator on L^X, there exists a unique fuzzy relation $R_F \in L^{X\times Y}$ such that $\langle R_F\rangle = F$ and $[R_F] = F^\partial$ where $R_F(x, y) = F(1_x)(y)$.

(3) $R_{\langle R\rangle} = R$.

Proof. (1) By Lemma 2.3, since

$$\begin{aligned} [R](\textstyle\bigwedge_{i\in\Gamma} A_i) &= \textstyle\bigwedge_{x\in X}(R(x, y) \to \bigwedge_{i\in\Gamma} A_i(x)) = \bigwedge_{i\in\Gamma} \bigwedge_{x\in X}(R(x, y) \to A_i(x)) \\ &= \textstyle\bigwedge_{i\in\Gamma}[R](A_i) \\ [R](\alpha \to A)(y) &= \textstyle\bigwedge_{x\in X}(R(x, y) \to (\alpha \to A)(y))) = \alpha \to [R](A)(y), \end{aligned}$$

then $[R]$ is a necessity operator. We easily show $\langle R\rangle$ is a modal operator. We have $\langle R\rangle = [R]^\partial$ from:

$$\begin{aligned} ([R](A^*))^*(y) &= \Big(\textstyle\bigwedge_{x\in X}(R(x, y) \to A^*(x))\Big)^* \\ &= \Big(\textstyle\bigwedge_{x\in X}(R(x, y) \odot A(x))^*\Big)^* \\ &= \textstyle\bigvee_{x\in X}(R(x, y) \odot A(x)) = \langle R\rangle(A)(y). \end{aligned}$$

(2) Since $A = \bigvee_{x\in X} A(x) \odot 1_x$ and F is a modal operator, then $F(A) = F(\bigvee_{x\in X}(A(x) \odot 1_x)) = \bigvee_{x\in X}(A(x) \odot F(1_x))$. Thus

$$\begin{aligned} \langle R_F\rangle(A)(y) &= \textstyle\bigvee_{x\in X}(R_F(x, y) \odot A(x)) = \bigvee_{x\in X}(F(1_x)(y) \odot A(x)) \\ &= F(\textstyle\bigvee_{x\in X} 1_x \odot A(x))(y) = F(A)(y). \end{aligned}$$

$$\begin{aligned} [R_F](A)(y) &= \textstyle\bigwedge_{x\in X}(R_F(x, y) \to A(x))) = \bigwedge_{x\in X}(F(1_x)(y) \to A(x))) \\ &= \Big(\textstyle\bigvee_{x\in X}(F(1_x)(y) \odot A^*(x))\Big)^* = \Big(F(\bigvee_{x\in X}(1_x(y) \odot A^*(x))\Big)^* \\ &= (F(A^*))^* = F^\partial(A). \end{aligned}$$

(3) $R_{\langle R\rangle}(x,y)=\langle R\rangle(1_x)(y)=\bigvee_{z\in X}(R(z,y)\odot 1_x(z))=R(x,y)$.

Theorem 3.7 *Let $R\in L^{X\times Y}$ be a fuzzy relation.*

(1) *$\langle\langle R\rangle\rangle^c$ is a modal operator and $[[R]]^c$ is a necessity operator with $\langle\langle R\rangle\rangle^c(A)=([[R]](A^*))^*=[[R]]^\partial(A)$ for each $A\in L^X$.*

(2) *If $F:L^X\to L^Y$ is a modal operator on L^X, there exists a unique fuzzy relation $R_F\in L^{X\times Y}$ such that $\langle\langle R_F\rangle\rangle^c=F$ and $[[R_F]]^c=F^\partial$ where $R_F(x,y)=F(1_x)^*(y)$.*

(3) *$R_{\langle\langle R\rangle\rangle^c}=R$.*

Proof. (1) Since $[[R]]^c(\bigwedge_{i\in\Gamma}A_i)(y)=\bigwedge_{x\in X}\Big((\bigwedge_{i\in\Gamma}A_i(x))^*\to R(x,y)\Big)=\bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(A_i^*(x)\to R(x,y))=\bigwedge_{i\in\Gamma}[[R]]^c(A_i)(y)$ and $[R](\alpha\to A)(y)=\bigwedge_{x\in X}((\alpha\to A)^*(x)\to R(x,y)))=\bigwedge_{x\in X}(R(x,y)^*\to(\alpha\to A)(x))=\alpha\to[[R]]^c(A)(y)$, $[[R]]^c$ is a necessity operator. We easily show $\langle\langle R\rangle\rangle^c$ is a modal operator. We have $\langle\langle R\rangle\rangle^c(A)=([[R]](A^*))^*=[[R]]^\partial(A)$ from:

$$\begin{aligned}([[R]]^c(A^*))^*(y)&=\Big(\bigwedge_{x\in X}(A(x)\to R(x,y))\Big)^*\\&=\Big(\bigwedge_{x\in X}(R^*(x,y)\odot A(x))^*\Big)^*\\&=\bigvee_{x\in X}(R^*(x,y)\odot A(x))=\langle\langle R\rangle\rangle^c(A)(y).\end{aligned}$$

(2) Since $A=\bigvee_{x\in X}A(x)\odot 1_x$ and $F(A)=\bigvee_{x\in X}A(x)\odot F(1_x)$, we have

$$\begin{aligned}\langle\langle R_F\rangle\rangle^c(A)(y)&=\bigvee_{x\in X}(R_F^*(x,y)\odot A(x))\\&=\bigvee_{x\in X}(F(1_x)(y)\odot A(x))\\&=F(\bigvee_{x\in X}1_x\odot A(x))(y)\\&=F(A)(y).\end{aligned}$$

$$\begin{aligned}[[R_F]]^c(A)(y)&=\bigwedge_{x\in X}(A^*(x)\to R_F(x,y)))\\&=\bigwedge_{x\in X}(R_F^*(x,y)\to A(x)))\ \text{(by Lemma 2.3 (10))}\\&=\bigwedge_{x\in X}(F(1_x)(y)\to A(x)))\\&=\Big(\bigvee_{x\in X}(F(1_x)(y)\odot A^*(x))\Big)^*\\&=\Big(F(\bigvee_{x\in X}(1_x(y)\odot A^*(x))\Big)^*=(F(A^*))^*=F^\partial(A).\end{aligned}$$

(3) $R_{\langle\langle R\rangle\rangle^c}(x,y)=\langle\langle R\rangle\rangle^c(1_x)^*(y)=\Big(\bigvee_{z\in X}(R^*(z,y)\odot 1_x(z))\Big)^*=R(x,y)$.

Theorem 3.8 *Let $R\in L^{X\times Y}$ be a fuzzy relation.*

(1) *$[[R]]$ is a sufficiency operator and $\langle\langle R\rangle\rangle$ is a co-sufficiency operator with $\langle\langle R\rangle\rangle(A)=([[R]](A^*))^*=[[R]]^\partial(A)$ for each $A\in L^X$.*

(2) *If $F:L^X\to L^Y$ is a sufficiency operator on L^X, there exists a unique fuzzy relation $R_F\in L^{X\times Y}$ such that $[[R_F]]=F$ and $\langle\langle R_F\rangle\rangle=F^\partial$ where $R_F(x,y)=F(1_x)(y)$.*

(3) *$R_{[[R]]}=R$.*

Proof. (1) Since $\langle\langle R\rangle\rangle^c$ is a modal operator and $[[R]]^c$ is a necessity operator, by Theorem, $[[R]]$ is a sufficiency operator and $\langle\langle R\rangle\rangle$ is a co-sufficiency operator. We have $\langle\langle R\rangle\rangle(A) = ([[R]](A^*))^* = [[R]]^\partial(A)$ from:

$$\begin{aligned}([[R]](A^*))^*(x) &= \Big(\bigwedge_{y\in X}(A^*(y)\to R(x,y))\Big)^* \\ &= \Big(\bigwedge_{y\in X}(R^*(x,y)\odot A^*(y))^*\Big)^* \\ &= \bigvee_{y\in X}(R^*(x,y)\odot A^*(y)) = \langle\langle R\rangle\rangle(A)(x)\end{aligned}$$

(2) Since $F(\bigvee_{x\in X}(A(x)\odot 1_x)) = \bigwedge_{x\in X}(A(x)\to F(1_x)$, we have

$$\begin{aligned}[[R_F]](A)(y) &= \bigwedge_{x\in X}(A(x)\to R_F(x,y)) = \bigwedge_{x\in X}(A(x)\to F(1_x)(y)) \\ &= \bigwedge_{x\in X}(F(A(x)\odot 1_x)(y)) = F(\bigvee_{x\in X}(A(x)\odot 1_x))(y) = F(A)(y).\end{aligned}$$

$$\begin{aligned}\langle\langle R_F\rangle\rangle(A)(y) &= \bigvee_{x\in X}(R_F^*(x,y)\odot A^*(x)) = \bigvee_{x\in X}(F(1_x)^*(y)\odot A^*(x)) \\ &= \Big(\bigwedge_{x\in X}(A^*(x)\to F(1_x)(y))\Big)^* \\ &= \Big(F(\bigvee_{x\in X} 1_x\odot A^*(x))(y)\Big)^* = F^\partial(A)(y).\end{aligned}$$

(3) $R_{[[R]]}(x,y) = [[R]](1_x)(y) = \bigwedge_{z\in X}(1_x(z)\to R(z,y)) = R(x,y)$.

Theorem 3.9 *Let $R\in L^{X\times Y}$ be a fuzzy relation.*

(1) *$[R]^c$ is a sufficiency operator and $\langle R\rangle^c$ is a co-sufficiency operator with $[R]^c(A) = (\langle R\rangle^c(A^*))^*$.*

(2) *If $F : L^X\to L^Y$ is a sufficiency operator on L^X, there exists a unique fuzzy relation $R_F\in L^{X\times Y}$ such that $[R_F]^c = F$ and $\langle R_F\rangle^c = F^\partial$ where $R_F(x,y) = F(1_x)^*(y)$.*

(3) *$R_{[R]^c} = R$.*

Proof. (1) Since $[R]$ is a necessity operator and $\langle R\rangle$ is a modal operator, by Theorem , $[R]^c$ is a sufficiency operator and $\langle R\rangle^c$ is a co-sufficiency operator. We have $[R]^c(A) = (\langle R\rangle^c(A^*))^*$ from

$$\begin{aligned}(\langle R\rangle^c(A^*))^*(x) &= \Big(\bigvee_{x\in X}(A(x)\odot R(x,y))\Big)^* \\ &= \bigwedge_{y\in X}(R(x,y)\to A^*(x)) \\ &= [R]^c(A)(x)\end{aligned}$$

(2)

$$\begin{aligned}[R_F]^c(A)(y) &= \bigwedge_{x\in X}(R_F(x,y)\to A^*(x)) \\ &= \bigwedge_{x\in X}(A(x)\to F(1_x)(y)) \\ &= \bigwedge_{x\in X}(F(A(x)\odot 1_x)(y)) \\ &= F(\bigvee_{x\in X}(A(x)\odot 1_x))(y) = F(A)(y)\end{aligned}$$

$$\begin{array}{ll}\langle R_F\rangle^c(A)(y) & = \bigvee_{x\in X}(R_F(x,y)\odot A^*(x)) \\ & = \bigvee_{x\in X}(F(1_x)^*(y)\odot A^*(x)) \\ & = \Big(\bigwedge_{x\in X}(A^*(x)\to F(1_x)(y))\Big)^* \\ & = \Big(F(\bigvee_{x\in X} 1_x\odot A^*(x))(y)\Big)^* \\ & = F^\partial(A)(y)\end{array}$$

(3) $R_{[R]^c}(x,y) = [R]^c(1_x)^*(y) = \Big(\bigwedge_{z\in X}(R(z,y)\to 1_x^*(z))\Big)^* = R(x,y).$

Theorem 3.10 *Let $R \in L^{X\times Y}$ be a fuzzy relation.*

(1) *If $F : L^X \to L^Y$ is a necessity operator on L^X, there exists a unique fuzzy relation $R_F \in L^{X\times Y}$ such that $[R_F] = F$ and $\langle R_F\rangle = F^\partial$ where $R_F(x,y) = F(1_x^*)^*(y)$.*

(2) $R_{[R]} = R$.

Proof. (1)

$$\begin{array}{ll}[R_F](A)(y) & = \bigwedge_{x\in X}(R_F(x,y)\to A(x)) \\ & = \bigwedge_{x\in X}(F(1_x^*)^*(y)\to A(x)) \\ & = \bigwedge_{x\in X}(A^*(x)\to F(1_x^*)(y)) \\ & = \bigwedge_{x\in X} F(\bigwedge_{x\in X}(A^*(x)\to 1_x^*(y))) \\ & = F((\bigvee_{x\in X} A^*(x)\odot 1_x)^*)(y) \\ & = F((A^*)^*)(y) = F(A)(y)\end{array}$$

$$\begin{array}{ll}\langle R_F\rangle(A)(x) & = \bigvee_{x\in X}(R_F(x,y)\odot A(x)) \\ & = \bigvee_{x\in X}(F(1_x^*)^*(y)\odot A(x)) \\ & = \Big(\bigwedge_{x\in X}(A(x)\to F(1_x^*)(y))\Big)^* \\ & = \Big(F((\bigvee_{x\in X} 1_x\odot A(x))^*(y)\Big)^* \\ & = F(A^*)^*(y) = F^\partial(A)(y)\end{array}$$

(3) $R_{[R]}(x,y) = [R](1_x^*)^*(y) = \Big(\bigwedge_{z\in X}(R(z,y)\to 1_x^*(z))\Big)^* = R(x,y).$

Theorem 3.11 *Let $R \in L^{X\times Y}$ be a fuzzy relation.*

(1) *If $F : L^X \to L^Y$ is a co-sufficiency operator on L^X, there exists a unique fuzzy relation $R_F \in L^{X\times Y}$ such that $\langle\langle R_F\rangle\rangle = F$ and $[[R_F]] = F^\partial$ where $R_F(x,y) = F(1_x^*)^*(y)$.*

(2) $R_{\langle\langle R_F\rangle\rangle} = R$.

Proof. (1)

$$\begin{aligned}\langle\langle R_F\rangle\rangle(A)(x) &= \bigvee_{x\in X}(R_F^*(x,y)\odot A^*(x))\\ &= \bigvee_{x\in X}(F(1_x^*)(y)\odot A^*(x))\\ &= F(\bigwedge_{x\in X}(A^*(x)\to 1_x^*(y))\\ &= F((\bigvee_{x\in X} A^*(x)\odot 1_x)^*)(y)\\ &= F((A^*)^*)(y) = F(A)(y)\end{aligned}$$

$$\begin{aligned}[[R_F]](A)(y) &= \bigwedge_{x\in X}(A(x)\to R_F(x,y))\\ &= \bigwedge_{x\in X}(A(x)\to F(1_x^*)^*(y))\\ &= \Big(\bigvee_{x\in X}(A(x)\odot F(1_x^*)(y))\Big)^*\\ &= \Big(F(\bigwedge_{x\in X}(A(x)\to 1_x^*))(y)\Big)^*\\ &= \Big(F((\bigvee_{x\in X} 1_x\odot A(x))^*(y)\Big)^*\\ &= F(A^*)^*(y) = F^\partial(A)(y)\end{aligned}$$

(2) $R_{\langle\langle R\rangle\rangle}(x,y) = \langle\langle R\rangle\rangle(1_x^*)^*(y) = \Big(\bigvee_{z\in X}(R^*(z,y)\odot 1_x(z))\Big)^* = R(x,y).$

Theorem 3.12 *Let $R\in L^{X\times Y}$ be a fuzzy relation.*

(1) *If $F: L^X\to L^Y$ is a necessity operator on L^X, there exists a unique fuzzy relation $R_F\in L^{X\times Y}$ such that $[[R_F]]^c = F$ and $\langle\langle R_F\rangle\rangle^c = F^\partial$ where $R_F(x,y) = F(1_x^*)(y)$.*

(2) $R_{[[R_F]]^c} = R$.

Proof. (1) Since $A = \bigwedge_{x\in X}(A^*(x)\to 1_x^*)$, we have:

$$\begin{aligned}[[R_F]]^c(A)(y) &= \bigwedge_{x\in X}(A^*\to R_F(x,y))\\ &= \bigwedge_{x\in X}(A^*(x)\to F(1_x^*)(y))\\ &= F(\bigwedge_{x\in X}(A^*(x)\to 1_x^*))(y)\\ &= F(A)(y).\end{aligned}$$

Since $A^* = \bigwedge_{x\in X}(A(x)\to 1_x^*)$, we have:

$$\begin{aligned}\langle\langle R_F\rangle\rangle^c(A)(y) &= \bigvee_{x\in X}(R_F^*(x,y)\odot A(x))\\ &= \bigvee_{x\in X}(F(1_x^*)^*(y)\odot A(x))\\ &= \Big(\bigwedge_{x\in X}(A(x)\to F(1_x^*)(y))\Big)^*\\ &= \Big(F(\bigwedge_{x\in X}(A(x)\to 1_x^*))(y)\Big)^*\\ &= F(A^*)^*(y) = F^\partial(A)(y).\end{aligned}$$

(3) $R_{[[R_F]]^c}(x,y) = [[R_F]]^c(1_x^*)(y) = \bigwedge_{z\in X}(1_x(z)\to R(z,y)) = R(x,y).$

Theorem 3.13 *Let $R \in L^{X\times Y}$ be a fuzzy relation.*

(1) If $F : L^X \to L^Y$ is a co-sufficiency operator on L^X, there exists a unique fuzzy relation $R_F \in L^{X\times Y}$ such that $\langle R_F\rangle^c = F$ and $[R_F]^c = F^\partial$ where $R_F(x,y) = F(1^*_x)(y)$.

(2) $R_{\langle R_F\rangle^c} = R$.

Proof. (1)

$$\begin{aligned}
\langle R_F\rangle^c(y) &= \bigvee_{x\in X}(R_F(x,y) \odot A^*(x)) \\
&= \bigvee_{x\in X}(F(1^*_x)(y) \odot A^*(x)) \\
&= F(\bigwedge_{x\in X}(A^*(x) \to 1^*_x))(y) \\
&= F(A)(y)
\end{aligned}$$

$$\begin{aligned}
[R_F]^c(A)(y) &= \bigwedge_{x\in X}(R_F(x,y) \to A^*(x)) \\
&= \Big(\bigvee_{x\in X}(A(x) \odot F(1^*_x)(y))\Big)^* \\
&= \Big(F(\bigwedge_{x\in X}(A(x) \to 1^*_x))(y)\Big)^* \\
&= F(A^*)^*(y) = F^\partial(A)(y)
\end{aligned}$$

(2) $R_{\langle R_F\rangle^c}(x,y) = \langle R_F\rangle^c(1^*_x)(y) = \bigvee_{z\in X}(R(z,y) \odot 1_x(z)) = R(x,y)$.

Example 3.14 Let $X = \{a,b,c\}$ and $Y = \{x,y\}$ be a set and $(L = [0,1], \odot)$ with $a \odot b = \max\{0, a+b-1\}$ and $a \to b = \min\{1, 1-a+b\}$. Define $F, G : L^X \to L^Y$ as

$$F(1_a)(x) = 0.6, F(1_b)(x) = 0.2, F(1_c)(x) = 0.7$$

$$F(1_a)(y) = 0.5, F(1_b)(y) = 1.0, F(1_c)(y) = 0.9$$

$$G(1^*_a)^*(x) = 0.8, G(1^*_b)^*(x) = 0.6, G(1^*_c)^*(x) = 0.5$$

$$G(1^*_a)^*(y) = 0.7, G(1^*_b)^*(y) = 0.9, G(1^*_c)^*(y) = 0.4$$

(1) If F is a modal operator, then, by Theorem 3.6,

$$R_F = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 1.0 \\ 0.7 & 0.9 \end{pmatrix}$$

$$\langle R_F\rangle(A) = F(A) = \begin{pmatrix} 0.6 \odot A(a) \vee 0.2 \odot A(b) \vee 0.7 \odot A(c) \\ 0.5 \odot A(a) \vee A(b) \vee 0.9 \odot A(c) \end{pmatrix}$$

$$[R_F](A) = F^\partial(A) = \begin{pmatrix} (0.6 \to A(a)) \wedge (0.2 \to A(b)) \wedge (0.7 \to A(c)) \\ (0.5 \to A(a)) \wedge (1.0 \to A(b)) \wedge (0.9 \to A(c)) \end{pmatrix}$$

(2) If F is a modal operator, then, by Theorem 3.7,

$$R_F = \begin{pmatrix} 0.4 & 0.5 \\ 0.8 & 0.0 \\ 0.3 & 0.1 \end{pmatrix}$$

$$\langle\langle R_F \rangle\rangle^c(A) = F(A) = \begin{pmatrix} 0.6 \odot A(a) \vee 0.2 \odot A(b) \vee 0.7 \odot A(c) \\ 0.5 \odot A(a) \vee A(b) \vee 0.9 \odot A(c) \end{pmatrix}$$

Since $[[R_F]]^c(A)(y) = \bigwedge_{x\in X}(R^*(x,y) \to A(x))$, we have

$$[[R_F]]^c(A) = F^\partial(A) = \begin{pmatrix} (0.6 \to A(a)) \wedge (0.2 \to A(b)) \wedge (0.7 \to A(c)) \\ (0.5 \to A(a)) \wedge (1.0 \to A(b)) \wedge (0.9 \to A(c)) \end{pmatrix}$$

(3) If F is a sufficiency operator, then, by Theorem 3.8,

$$R_F = \begin{pmatrix} 0.6 & 0.5 \\ 0.2 & 1.0 \\ 0.7 & 0.9 \end{pmatrix}$$

$$[[R_F]](A) = F(A) = \begin{pmatrix} (A(a) \to 0.6) \wedge (A(b) \to 0.2) \wedge (A(c) \to 0.7) \\ (A(a) \to 0.5) \wedge (A(c) \to 0.9) \end{pmatrix}$$

$$\langle\langle R_F \rangle\rangle(A) = F^\partial(A) = \begin{pmatrix} 0.4 \odot A^*(a) \vee 0.8 \odot A^*(b) \vee 0.3 \odot A^*(c) \\ 0.5 \odot A^*(a) \vee 0.1 \odot A^*(c) \end{pmatrix}$$

(4) If F is a sufficiency operator, then, by Theorem 3.9,

$$R_F = \begin{pmatrix} 0.4 & 0.5 \\ 0.8 & 0.0 \\ 0.3 & 0.1 \end{pmatrix}$$

$$[R_F]^c(A) = F(A) = \begin{pmatrix} (A(a) \to 0.6) \wedge (A(b) \to 0.2) \wedge (A(c) \to 0.7) \\ (A(a) \to 0.5) \wedge (A(c) \to 0.9) \end{pmatrix}$$

$$\langle R_F \rangle^c(A) = F^\partial(A) = \begin{pmatrix} 0.4 \odot A^*(a) \vee 0.8 \odot A^*(b) \vee 0.3 \odot A^*(c) \\ 0.5 \odot A^*(a) \vee 0.1 \odot A^*(c) \end{pmatrix}$$

(5) If G is a necessity operator, then, by Theorem 3.10,

$$R_G = \begin{pmatrix} 0.8 & 0.7 \\ 0.6 & 0.9 \\ 0.5 & 0.4 \end{pmatrix}$$

$$[R_G](A) = G(A) = \begin{pmatrix} (0.8 \to A(a)) \wedge (0.6 \to A(b)) \wedge (0.5 \to A(c)) \\ (0.7 \to A(a)) \wedge (0.9 \to A(b)) \wedge (0.4 \to A(c)) \end{pmatrix}$$

$$\langle R_G\rangle(A) = G^{\partial}(A) = \begin{pmatrix} 0.8 \odot A(a) \vee 0.4 \odot A(b) \vee 0.5 \odot A(c) \\ 0.7 \odot A(a) \vee 0.9 \odot A(b) \vee 0.4 \odot A(c) \end{pmatrix}$$

(6) If G is a necessity operator, then, by Theorem 3.12,

$$R_G = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \\ 0.5 & 0.6 \end{pmatrix}$$

$$[[R_G]]^c(A) = G(A) = \begin{pmatrix} (0.8 \to A(a)) \wedge (0.6 \to A(b)) \wedge (0.5 \to A(c)) \\ (0.7 \to A(a)) \wedge (0.9 \to A(b)) \wedge (0.4 \to A(c)) \end{pmatrix}$$

$$\langle\langle R_G\rangle\rangle^c(A) = G^{\partial}(A) = \begin{pmatrix} 0.8 \odot A(a) \vee 0.4 \odot A(b) \vee 0.5 \odot A(c) \\ 0.7 \odot A(a) \vee 0.9 \odot A(b) \vee 0.4 \odot A(c) \end{pmatrix}$$

(7) If G is a co-sufficiency operator, then $G(A) = G(\bigwedge_{x\in X}(A^*(x) \to 1_x^*)) = \bigvee_{x\in X}(A^*(x) \odot G(1_x^*)$. By Theorem 3.11, we have:

$$R_G = \begin{pmatrix} 0.8 & 0.7 \\ 0.6 & 0.9 \\ 0.5 & 0.4 \end{pmatrix}$$

$$\langle\langle R_G\rangle\rangle(A) = G(A) = \begin{pmatrix} (0.2 \odot A^*(a)) \vee (0.4 \odot A^*(b)) \vee (0.5 \odot A^*(c)) \\ (0.3 \odot A^*(a)) \vee (0.1 \odot A^*(b)) \vee (0.6 \odot A^*(c)) \end{pmatrix}$$

$$[[R_G]](A) = G^{\partial}(A) = \begin{pmatrix} (A(a) \to 0.8) \wedge (A(b) \to 0.6) \wedge (A(c) \to 0.5) \\ (A(a) \to 0.7) \wedge (A(b) \to 0.9) \wedge (A(c) \to 0.4) \end{pmatrix}$$

(8) If G is a co-sufficiency operator, then by Theorem 3.13, we have:

$$R_G = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \\ 0.5 & 0.6 \end{pmatrix}$$

$$\langle R_G\rangle^c(A) = G(A) = \begin{pmatrix} (0.2 \odot A^*(a)) \vee (0.4 \odot A^*(b)) \vee (0.5 \odot A^*(c)) \\ (0.3 \odot A^*(a)) \vee (0.1 \odot A^*(b)) \vee (0.6 \odot A^*(c)) \end{pmatrix}$$

$$[R_G]^c(A) = G^{\partial}(A) = \begin{pmatrix} (A(a) \to 0.8) \wedge (A(b) \to 0.6) \wedge (A(c) \to 0.5) \\ (A(a) \to 0.7) \wedge (A(b) \to 0.9) \wedge (A(c) \to 0.4) \end{pmatrix}$$

References

[1] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[2] Ivo. Düntsch, Ewa Orlowska, Boolean algebras arising from information systems, *Annals of Pure and Applied Logic,* **127** (2004), 77-98.

[3] P. Hájek, *Metamathematices of Fuzzy Logic,* Kluwer Academic Publishers, Dordrecht (1998).

[4] U. Höhle, E. P. Klement, *Non-classical logic and their applications to fuzzy subsets ,* Kluwer Academic Publisher, Boston, 1995.

[5] Y.C. Kim, Modal, necessity, sufficiency and co-sufficiency operators, *Korean J. Math.,* **20** (2012), 293-305.

[6] Y.C. Kim, Relations and modal operators, *Mathematica Aeterna*, **2(6)** (2012), 497-508.

[7] Ewa Orlowska, I. Rewitzky, Algebras for Galois-style connections and their discrete duality, *Fuzzy Sets and Systems*, **161** (2010), 1325-1342.

[8] A.M. Radzikowska, E.E. Kerre, Characterisation of main classes of fuzzy relations using fuzzy modal operators, *Fuzzy Sets and Systems*, **152** (2005), 223-247.

[9] R. Wille, *Restructuring lattice theory; an approach based on hierarchies of concept*, in: 1. Rival(Ed.), Ordered Sets, Reidel, Dordrecht, Boston, 1982, 445-470.

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