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Uniqueness type result in dimension 3.

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Abstract

We give some estimates of type $\sup \times \inf$ on Riemannian manifold of dimension 3 for a prescribed curvature type equation. As a consequence, we derive an uniqueness type result.

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1 Introduction and Main Results

In this paper, we deal with the following prescribed scalar curvature type equation in dimension 3:

$$\Delta u + h(x)u = V(x)u^5, \; u > 0. \qquad (E)$$

Where h, V are two continuous functions. In the case $8h = R_g$ the scalar curvature, we call V the prescribed scalar curvature. Here, we assume h a bounded function and $h_0 = ||h||_{L^\infty(M)}$.

We consider three positive real number a, b, A and we suppose V lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } ||\nabla V||_{L^\infty(M)} \leq A. \qquad (C)$$

The equation (E) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a $\sup \times \inf$ inequality.

The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u = V(x)e^u, \qquad (E')$$

The equation (E') was studed by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, \inf_\Omega u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [19], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω, C is a positive constant which depends on $\dfrac{\inf_\Omega V}{\sup_\Omega V}$, and, $\alpha \in (0, 1]$.

In the case $V \equiv 1$ and M compact, the equation (E) is Yamabe equation. Yamabe has tried to solve problem but he could not, see [22]. T.Aubin and R.Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary.

When M is a compact Riemannian manifold, there exist some compactness result for equation (E) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [6].

Our proof is an extension of Brezis-Li and Li-Zhang result in dimension 3, see [7] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method.

Here, we give an equality of type $\sup \times \inf$ for the equation (E) with general conditions (C). Note that, in our proof, we do not need a classification result for some particular elliptic PDEs on $\mathbb{R}^3$.

In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5].

There are other estimates of type sup + inf on complex Monge-Ampere equation on compact manifolds, see [20-21] . They consider, on compact Kahler manifold (M, g), the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\phi)^n = e^{f-t\phi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\phi > 0 \text{ on } M \end{cases}$$

And, they prove some estimates of type $\sup_M + m \inf_M \leq C$ or $\sup_M + m \inf_M \geq C$ under the positivity of the first Chern class of M.

Here, we have,

Theorem 1.1 *For all compact set K of M and all positive numbers a, b, A, h_0 there is a positive constant c, which depends only on, a, b, A, h_0, K, M, g such that:*

$$\sup_K u \times \inf_M u \leq c,$$

for all u solution of (E) with conditions (C).

This theorem generalise Li-Zhang result, see [17] in the case $V \equiv 1$. Here, we use Li and Zhang method in [17].

In the case $h \equiv \epsilon \in (0, 1)$ and u_ϵ solution of :

$$\Delta u_\epsilon + \epsilon u_\epsilon = V_\epsilon u_\epsilon^5, \ u_\epsilon > 0. \qquad (E_\epsilon)$$

We have:

Corollary 1.2 *For all compact set K of M and all positive numbers a, b, A there is a positive constant c, which depends only on, a, b, A, K, M, g such that:*

$$\sup_K u_\epsilon \times \inf_M u_\epsilon \leq c,$$

for all u solution of (E_ϵ) with conditions (C).

Now, if we assume M a compact riemannian manifold and $0 < a \leq V_\epsilon \leq b < +\infty$
we have:

Theorem 1.3 *(*see [3]*). For all positive numbers a, b, m there is a positive constant c, which depends only on, a, b, m, M, g such that:*

$$\epsilon \sup_M u_\epsilon \times \inf_M u_\epsilon \geq c,$$

for all u_ϵ solution of (E_ϵ) with

$$\max_M u_\epsilon \geq m > 0$$

.

As a consequence of the two previous theorems, we have:

Theorem 1.4 *For all positive numbers a, b, A we have:*

$$\max_M u_\epsilon \to 0,$$

and (up to a subsequence),

$$\frac{\max_M u_\epsilon}{\epsilon^{1/4}} \to w_0 > 0, \ \ \text{and}, \ \ \frac{\min_M u_\epsilon}{\epsilon^{1/4}} \to w_0 > 0.$$

Remarks:

- It is not necessary to have $u_\epsilon \equiv w_0 \epsilon^{1/4}$, because if we take a nonconsant function V, we can find by the variational approach a non constant positive solution of the subcritical equation:

$$\Delta u_\epsilon + \epsilon u_\epsilon = \mu_\epsilon V u_\epsilon^{5-\epsilon}, \ \text{with} \ \mu_\epsilon, u_\epsilon > 0.$$

  In this case (subcritical which tends to the critical) we also have the $\sup \times \inf$ inequalities and the uniqueness type theorem.

- In fact, we prove, up to a subsequence that $\dfrac{u_\epsilon}{\epsilon^{1/4}}$ converge to a constant which depends on a, b and A.

2 Proof of the theorems

Proof of theorem 1.1:

We want to prove that:

$$\epsilon \max_{B(0,\epsilon)} u \times \min_{B(0,4\epsilon)} u \leq c = c(a, b, A, M, g) \qquad (1)$$

We argue by contradiction and we assume that:

$$\max_{B(0,\epsilon_k)} u_k \times \min_{B(0,4\epsilon_k)} u_k \geq k {\epsilon_k}^{-1} \qquad (2)$$

Step 1: The blow-up analysis

The blow-up analysis gives us :
For some $\bar{x}_k \in B(0,\epsilon_k)$, $u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k$, and, from the hypothesis,

$$u_k(\bar{x}_k)^2 \epsilon_k \to +\infty.$$

By a standard selection process, we can find $x_k \in B(\bar{x}_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying,

$$u_k(x_k)^2 \sigma_k \to +\infty, \qquad (3)$$

$$u_k(x_k) \geq u_k(\bar{x}_k), \qquad (4)$$

and,

$$u_k(x) \leq C_1 u_k(x_k), \text{ in } B(x_k, \sigma_k), \qquad (5)$$

where C_1 is some universal constant.
It follows from above ((2), (4)) that:

$$u_k(x_k) \times \min_{\partial B(x_k, 2\epsilon_k)} u_k \epsilon_k \geq u_k(\bar{x}_k) \times \min_{B(0,4\epsilon_k)} u_k \epsilon_k \geq k \to +\infty. \qquad (6)$$

We use $\{z^1, \ldots, z^n\}$ to denote some geodesic normal coordinates centered at x_k (we use the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z) dz dz^j$,

$$g_{ij}(z) - \delta_{ij} = O(r^2), \; g := det(g_{ij}(z)) = 1 + O(r^2), \; h(z) = O(1), \qquad (7)$$

where $r = |z|$. Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_{ij} u,$$

where

$$b_j = O(r), \; d_{ij} = O(r^2) \qquad (8)$$

We have a new function:

$$v_k(y) = M_k^{-1} u_k(M_k^{-2} y) \text{ for } |y| \leq 3\epsilon_k M_k^2$$

where $M_k = u_k(0)$.
From (5), (6), we have:

$$\begin{cases} \Delta v_k + \bar{b}_i \partial_i v_k + \bar{d}_{ij} \partial_{ij} v_k - \bar{c} v_k + {v_k}^5 = 0 \text{ for } |y| \leq 3\epsilon_k M_k^2 \\ v_k(0) = 1 \\ v_k(y) \leq C_1 \text{ for } |y| \leq \sigma_k M_k^2 \\ \lim_{k \to +\infty} \min_{|y| = 2\epsilon_k M_k^2} (v_k(y)|y|) = +\infty \qquad (9) \end{cases}$$

where C_1 is a universal constant and

$$\bar{b}_i(y) = M_k^{-2} b_i(M_k^{-2} y), \; \bar{d}_{ij}(y) = d_{ij}(M_k^{-2} y) \qquad (10)$$

and,

$$\bar{c}(y) = M_k^{-4} h(M_k^{-2} y) \qquad (11)$$

We can see that for $|y| \leq 3\epsilon_k M_k^2$, we have:

$$|\bar{b}_i(y)| \leq C M_k^{-4} |y|, \; |\bar{d}_{ij}(y)| \leq C M_k^{-4} |y|^2, \; |\bar{c}(y)| \leq C M_k^{-4} \qquad (12)$$

where C depends on n, M, g.

It follows from $(9), (10), (11), (12)$ and the elliptic estimates, that, along a subsequence, v_k converges in C^2 norm on any compact subset of $\mathbb{R}^2$ to a positive function U satisfying:

$$\begin{cases} \Delta U + U^5 = 0 \text{ in } \mathbb{R}^2 \\ U(0) = 1, \; 0 < U \leq C_1 \qquad (13) \end{cases}$$

<u>Step 2: The Kelvin transform and moving-plane method</u>

For $x \in \mathbb{R}^2$ and $\lambda > 0$, let,

$$v_k^{\lambda,x}(y) := \frac{\lambda}{|y-x|} v_k \left(x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right)$$

denote the Kelvin transformation of v_k with respect to the ball centered at x and of radius λ.

We want to compare for fixed x, v_k and $v_k^{\lambda,x}$. For simplicity we assume $x = 0$. We have:

$$v_k^\lambda(y) := \frac{\lambda}{|y|} v_k(y^\lambda), \text{ with } y^\lambda = \frac{\lambda^2 y}{|y|^2}$$

For $\lambda > 0$, we set,

$$\Sigma_\lambda = B\left(0, \epsilon_k {M_k}^2\right) - \bar{B}(0, \lambda).$$

The boundary condition, (9), become:

$$\lim_{k\to+\infty}\min_{|y|=\epsilon_k M_k^2}(v_k(y)|y|)=\lim_{k\to+\infty}\min_{|y|=2\epsilon_k M_k^2}(v_k(y)|y|)=+\infty \qquad (14)$$

We have:

$$\Delta v_k^\lambda + V_k^\lambda (v_k^\lambda)^5 = E_1(y) \text{ for } y\in\Sigma_\lambda \qquad (15)$$

where,

$$E_1(y)=-\left(\frac{\lambda}{|y|}\right)^5\left(\bar b_i(y^\lambda)\partial_i v_k(y^\lambda)+\bar d_{ij}(y^\lambda)\partial_{ij}v_k(y^\lambda)-\bar c(y^\lambda)v_k(y^\lambda)\right). \qquad (16)$$

Clearly, from $(10), (11)$, there exists $C_2 = C_2(\lambda_1)$ such that,

$$|E_1(y)|\le C_2\lambda^5 M_k^{-4}|y|^{-5} \text{ for } y\in\Sigma_\lambda \qquad (17)$$

Let,

$$w_\lambda = v_k - v_k^\lambda.$$

Here, we have, for simplicity, omitted k. We observe that by $(9), (15)$:

$$\Delta w_\lambda + \bar b_i\partial_i w_\lambda + \bar d_{ij}\partial_{ij} w_\lambda - \bar c w_\lambda + 5\xi^4 V_k w_\lambda = E_\lambda \text{ in } \Sigma_\lambda \qquad (18)$$

where ξ stay between v_k and v_k^λ, and,

$$E_\lambda = -\bar b_i\partial_i v_k^\lambda + \bar d_{ij}\partial_{ij}v_k^\lambda + \bar c v_k^\lambda - E_1 - (V_k - V_k^\lambda)(v_k^\lambda)^5. \qquad (19)$$

A computations give us the following two estimates:

$$|\partial_i v_k^\lambda(y)|\le C\lambda|y|^{-2}, \text{ and } |\partial_{ij} v_k^\lambda(y)|\le C\lambda|y|^{-3} \text{ in } \Sigma_\lambda \qquad (20)$$

From $(10), (11), (20)$, we have,

Lemma 2.1 . *For somme constant* $C_3 = C_3(\lambda)$

$$|E_\lambda|\le C_3\lambda M_k^{-4}|y|^{-1}+C_3\lambda^5 M_k^{-2}|y|^{-4} \text{ in } \Sigma_\lambda \qquad (21)$$

we consider the following auxiliary function:

$$h_\lambda = -C_1 A M_k^{-2}\lambda^2\left(1-\frac{\lambda}{|y|}\right)+C_2 A M_k^{-2}\lambda^3\left(1-\left(\frac{\lambda}{|y|}\right)^2\right)-C_3 M_k^{-4}\lambda(|y|-\lambda),$$

where C_1, C_2 and C_3 are three positive numbers.

Lemma 2.2 . *We have,*

$$w_\lambda + h_\lambda \geq 0, \text{ in } \Sigma_\lambda \ \forall 0 < \lambda \leq \lambda_1 \qquad (22)$$

Proof of Lemma 2.2. We divide the proof into two steps.

Step 1. There exists $\lambda_{0,k} > 0$ such that (22) holds :

$$w_\lambda + h_\lambda \geq 0, \text{ in } \Sigma_\lambda \ \forall 0 < \lambda \leq \lambda_{0,k}.$$

To see this, we write:

$$w_\lambda = v_k(y) - v_k^\lambda(y) = \frac{1}{\sqrt{|y|}} \left(\sqrt{|y|} v_k(y) - \sqrt{|y^\lambda|} v_k(y^\lambda) \right).$$

Note that y and y^λ are on the same ray starting from the origin. Let, in polar coordinates,

$$f(r, \theta) = \sqrt{r} v_k(r, \theta).$$

From the uniform convergence of v_k, there exists $r_0 > 0$ and $C > 0$ independant of k such that,

$$\frac{\partial f}{\partial r}(r, \theta) > C r^{-1/2} \text{ for } 0 < r < r_0.$$

Consequently, for $0 < \lambda < |y| < r_0$, we have:

$$w_\lambda(y) + h_\lambda(y) = v_k(y) - v_k^\lambda(y) + h_\lambda(y),$$

$$> \frac{1}{\sqrt{r_0}} C \sqrt{r_0}^{-1/2} (|y| - |y^\lambda|) + h_\lambda(y)$$

$$> (\frac{C}{r_0} - C_3 \lambda M_k^{-2})(|y| - \lambda) \text{ since } |y| - |y^\lambda| > |y| - \lambda$$

$$> 0. \qquad (23)$$

Since,

$$|h_\lambda(y)| + v_k^\lambda(y) \leq C(k, r_0)\lambda, \ \ r_0 \leq |y| \leq \epsilon_k {M_k}^2,$$

we can pick small $\lambda_{0,k} \in (0, r_0)$ such that for all $0 < \lambda \leq \lambda_{0,k}$ we have,

$$w_\lambda(y) + h_\lambda(y) \geq \min_{B(0, \epsilon_k M_k^2)} v_k - C(k, r_0)\lambda_{0,k} > 0 \ \forall \ r_0 \leq |y| \leq \epsilon_k {M_k}^2$$

Step 1 follows from (23).

Let,

$$\bar{\lambda}^k = \sup\{0 < \lambda \leq \lambda_1, w_\mu + h_\mu \geq 0, \text{ in } \Sigma_\mu \ \forall 0 < \mu \leq \lambda\} \qquad (24)$$

Step 2. $\bar{\lambda}^k = \lambda_1$, (22) holds.
For this, the main estimate needed is:

$$(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c} + 5\xi^4 V_k)(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda \qquad (25)$$

Thus,

$$\Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij}\partial_{ij} h_\lambda + (-\bar{c} + 5\xi^4 V_k) h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda. \qquad (26)$$

We have $h_\lambda < 0$, and, (12) and a computation give us,

$$|\bar{c} h_\lambda| \leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^2 M_k^{-6} \leq C_3 \lambda M_k^{-4} |y|^{-1},$$

and,

$$|\bar{b}_i \partial_i h_\lambda| + |\bar{d}_{ij}\partial_{ij} h_\lambda| \leq C_3 \lambda M_k^{-8} |y| + C_3 \lambda^3 M_k^{-6} |y|^{-1} + C_3 \lambda^5 M_k^{-6} |y|^{-2},$$

$$\leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4}$$

Thus,

$$|\bar{b}_i \partial_i h_\lambda| + |\bar{d}_{ij}\partial_{ij} h_\lambda| + |\bar{c} h_\lambda| \leq C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4} \text{ in } \Sigma_\lambda$$

Thus, by (21),

$$\Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij}\partial_{ij} h_\lambda + (-\bar{c} + 5\xi^4 V_k) h_\lambda + E_\lambda \leq$$

$$\leq \Delta h_\lambda + C_3 \lambda M_k^{-4} |y|^{-1} + C_3 \lambda^5 M_k^{-2} |y|^{-4} + |E_\lambda| \leq 0,$$

because,

$$\Delta h_\lambda = -2C_3 \lambda M_k^{-4} |y|^{-1} - 2C_3 \lambda^5 M_k^{-2} |y|^{-4}.$$

From the boundary condition and the definition of v_k^λ and h_λ, we have:

$$|h_\lambda(y)| + v_k^\lambda(y) \leq \frac{C(\lambda_1)}{|y|}, \quad \forall \ |y| = \epsilon_k {M_k}^2,$$

and, thus,

$$w_{\bar\lambda^k}(y) + h_{\bar\lambda^k}(y) > 0 \ \ \forall\ |y| = \epsilon_k {M_k}^2,$$

We can use the maximum principal and the Hopf lemma to have:

$$w_{\bar\lambda^k} + h_{\bar\lambda^k} > 0, \text{ in } \ \Sigma_\lambda,$$

and,

$$\frac{\partial}{\partial\nu}(w_{\bar\lambda^k} + h_{\bar\lambda^k}) > 0, \text{ in } \ \Sigma_\lambda.$$

From (25) and above we conclude that $\bar\lambda^k = \lambda_1$ and lemma 2.2 is proved.

Given any $\lambda > 0$, since the sequence v_k converges to U and $h_{\bar\lambda^k}$ converges to 0 on any compact subset of $\mathbb{R}^2$, we have:

$$U(y) \geq U^\lambda(y), , \ \ \forall\ |y| \geq \lambda, \ \forall\ 0 < \lambda < \lambda_1.$$

Since $\lambda_1 > 0$ is arbitrary, and since we can apply the same argument to compare v_k and $v_k^{\lambda,x}$, we have:

$$U(y) \geq U^{\lambda,x}(y), , \ \ \forall\ |y-x| \geq \lambda > 0.$$

Thus implies that U is a constant which is a contradiction.

Proof of theorem 1.4:

From theorem 2.1 (see [3]), we have:

$$\max_M u_\epsilon \to 0. \qquad (27)$$

We conclude with the aid of the elliptic estimates and the classical Harnack inequality that:

$$\max_M u_\epsilon \leq C \min_M u_\epsilon, \qquad (28)$$

where C is a positive constant independant of ϵ.

Let G_ϵ the Green function of the operator $\Delta + \epsilon$, we have,

$$\int_M G_\epsilon(x,y) dV_g(y) = \frac{1}{\epsilon}, \ \forall\ x \in M. \qquad (29)$$

We write:

$$\inf_M u_\epsilon = u_\epsilon(x_\epsilon) = \int_M G_\epsilon(x_\epsilon, y) V_\epsilon(y) u_\epsilon^5(y) dV_g(y) \geq$$

$$\geq a(\inf_M u_\epsilon)^5 \int_M G_\epsilon(x_\epsilon, y) dV_g(y) = a\frac{(\inf_M u_\epsilon)^5}{\epsilon},$$

thus,

$$\inf_M u_\epsilon \leq C_1 \epsilon^{1/4}. \qquad (30)$$

With the similar argument, we have :

$$\sup_M u_\epsilon \geq C_2 \epsilon^{1/4}. \qquad (31)$$

Finaly, we have:

$$C_1 \epsilon^{1/4} \leq u_\epsilon(x) \leq C_2 \epsilon^{1/4} \;\; \forall \, x \in M. \qquad (32)$$

Where C_1 and C_2 are two positive constant independant of ϵ.

We set $w_\epsilon = \dfrac{u_\epsilon}{\epsilon^{1/4}}$, then,

$$\Delta w_\epsilon + \epsilon w_\epsilon = \epsilon V_\epsilon w_\epsilon^5.$$

The theorem follow from the standard elliptic estimate, the Green function of the lapalcian and the Green representation formula for the solutions of the previous equation.

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