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Two types of traveling wave solutions of a KdV-like advection-dispersion equation

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Abstract

We present a KdV-like 2-parameter equation $u_t + (3(1-\delta)u + (\delta + 1)\frac{u_{xx}}{u_x})u_x = \epsilon u_{xxx}$. By using the dynamical system method, existence of different traveling wave solutions are discussed, including smooth solitary wave solution of with bell type, solitary wave solutions of valley type and peakon wave solution of valley type. Numerical integration are used to shown the different types of solutions.

Mathematics Subject Classification: 35Q51, 35Q58,37K50

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1 Introduction

Many nonlinear partial differential equations have been found to have a variety of traveling wave solutions. For instances, the well-known Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

has solitary wave solutions and its solitary waves are solitons [1]. Its extension, the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{2}$$

process peakon solutions[2]. Peakon solutions have a sharp peak with a discontinuous first derivative.

The KdV equation has purely linear dispersion. The KdV soliton is the balance between nonlinear steepening and linear dispersion. However, the CH equation introduces additional higher order combinations of nonlinear/nonlocal balance. Even in the limit of vanishing linear dispersion, nonlinear dynamics still remains and peakon solutions appear. There are abundant studies on classical solitons and special peakon solitons [3–7].

An equation related to KdV in a similar way, called SIdV equation

$$u_t + \left(\frac{u_{xx}}{u} u_x\right) = \epsilon u_{xxx} \tag{3}$$

was introduced in [8]. What is interesting is that the advecting velocity is a quotient $2u_{xx}/u$, not the linear form $6u$ in the KdV equation. There are special values of ϵ at which the SIdV comes close to the KdV equation. Despite of the different advecting velocity, the SIdV equation has the same solitary wave solution as the KdV equation.

The nonlinear advecting form has been studied by Qiao and Li [9]. They derived the following equation

$$(-\frac{u_{xx}}{u})_t = 2uu_x. \tag{4}$$

and pointed out that it has classical solitons, periodic soliton and kink solutions.

We are interesting in that whether there are special solitons solutions to other KdV-like equations with advection term $\frac{u_{xx}}{u}$. In this paper, we consider the following generalized SIdV equation

$$u_t + \left(3(1-\delta)u + (\delta+1)\frac{u_{xx}}{u}\right) u_x = \epsilon u_{xxx} \tag{5}$$

where δ and ϵ are constants. It can be thought as a nonlinear wave equation for the dispersive advection of the real wave amplitude u. Clearly it is a generalization of the KdV-SidV 1-parameter family in [**?**]. When $\delta = \epsilon = 1$, (5) turns to be the SIdV equation (4). So it interpolates between SIdV and KdV.

We look for traveling wave solutions of (5). We shall apply the bifurcation theory of dynamical systems [10] in this study.

The rest of the paper is organized as follows. Section 2 gives bifurcations conditions of (10) and different phase portraits associated with different parameters. Section 3 concerns the existence of smooth and non-smooth traveling wave solutions of (5).Section 4 is the conclusions.

2 Phase portraits of

We look for traveling wave solutions of (5) in the form of

$$u(x,t) = \phi(\xi), \quad \xi = x - ct, \tag{6}$$

where c is the wave speed. We only consider the situation $c > 0$. That means the wave traveling to the right. Substituting (6) into (5) then (5) is reduced to

$$-c\phi\phi' + 3(1-\delta)\phi^2\phi' + (1+\delta)\phi'\phi'' - \epsilon\,\phi\phi''' = 0 \tag{7}$$

where "$'$" is the derivative with respect to ξ. Integrating (7) once and setting the integrating constant as g, we get

$$2\,(1-\delta)\,\phi^3 - c\phi^2 - 2\epsilon\,\phi\phi'' + (1+\delta+\epsilon)\phi'^2 - 2\,g \tag{8}$$

Eq. (8) is equivalent to the planar system

$$\begin{aligned} \frac{d\phi}{d\xi} &= y \\ \frac{dy}{d\xi} &= \frac{(\delta+1+\epsilon)\,y^2 + 2(1-\delta)\phi^3 - c\phi^2 - 2\,g}{2\epsilon\phi} \end{aligned} \tag{9}$$

Since the phase orbits defined by the vector fields of system (9) determine all traveling wave solutions of (5), we shall investigate the bifurcations of the phase portraits of (9) in the phase plane as the parameters are changed. Here we only consider the bounded solutions.

Clearly, system (9) has a singular line $\phi = 0$. On the singular straight line of the phase plane (ϕ, y), ϕ'' has no definition. To avoid the singularity, let $d\xi = 2\epsilon\phi d\tau$ for $\phi \neq 0$. Then system (9) becomes an regular system

$$\begin{aligned} \frac{d\phi}{d\tau} &= 2\epsilon\phi y \\ \frac{dy}{d\tau} &= (\delta+1+\epsilon)\,y^2 + 2\,\phi^3 - 2\,\phi^3\delta - \phi^2 c - 2\,g \end{aligned} \tag{10}$$

Both (9) and (10) have the following first integral

$$y^2 - 2\,\frac{-1+\delta}{\delta+1-2\,\epsilon}\phi^3 - \frac{c}{\delta+1-\epsilon}\phi^2 - 2\,\frac{g}{\delta+1+\epsilon} = h\phi^{\frac{\delta+1+\epsilon}{\epsilon}}, \tag{11}$$

where h is an arbitrary constant.

We now investigate the bifurcations of phase portraits of system (10). Let

$$f(\phi) = 2(1-\delta)\phi^3 - c\phi^2 - 2g$$

Then $f'(\phi) = 2\phi(3(1-\delta)\phi - c)$ has two roots $\phi_1^* = 0$ and $\phi_2^* = \frac{c}{3(1-\delta)}$ provided $c^2 - 12(1-\delta) > 0$. It follows that $f''(\phi_1^*) = -2c$, $f''(\phi_2^*) = 2c$. Then $f(\phi_1^*) = -2g$ is a local maximum value while $f(\phi_2^*) = -2g - \frac{c^3}{27(1-\delta)^2}$ is a local minimum value. Without loss of generality, we assume $0 < \delta < 1$. Then we can see that $\phi_1^* < \phi_2^*$.

On the $\phi-$axis, there are at most three equilibrium points E_1, E_2 and E_3 for (10) . There are two equilibrium points $E_s^{\pm}(0, Y_s^{\pm})$ where $Y_s^{\pm} = \pm\sqrt{\frac{2g}{\delta+1+\epsilon}}$ when $\frac{2g}{\delta+1+\epsilon} > 0$. Without loss of generality, we assume $\delta > 0$ and $\epsilon > 0$.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (10) at an equilibrium point (ϕ_e, y_e). Then we have

$$\begin{aligned} &J(\phi_i, 0) = detM(\phi_i, 0) = -2\epsilon\phi_i f'(\phi_i), \\ &J(0, Y_s^{\pm}) = detM(0, Y_s^{\pm}) = 4\epsilon(1+\epsilon+\delta)Y_s^{\pm 2} > 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &p(\phi_i, 0) = traceM(\phi_i, 0) = 0, \\ &p(0, Y_s^{\pm}) = traceM(0, Y_s^{\pm}) = 2Y_s^{\pm}(1+2\epsilon+\delta), \\ &p^2 - 4J = 4(1+\delta)^2 Y_s^{\pm 2} > 0. \end{aligned} \tag{13}$$

By the theory of planar dynamical systems [10], for an equilibrium point of a planar integral system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0 = p$, then it is a center; if $p^2 > 4J > 0$, then it is a node (stable if $p < 0$, unstable if $p > 0$); if $J = 0$ and the Poincaré index of the equilibrium point is zero, then it is a cusp.

From (12) we see that the types of the equilibrium points $E_i(\phi_i, 0)$ of system (10) are determined by the sign of $f'(\phi_i)$ and the sign of ϕ_i.

When $f(\phi_1^*) = 0$ we get the parameter condition $g = 0$. Letting $f(\phi_2*) = 0$ we have the parameter condition $g^* = -\frac{c^3}{54(1-\delta)^2}$.

Under the parameter condition, it is clear that $lim_{\phi\to-\infty} = -\infty$ and $lim_{\phi\to\infty} = \infty$. The function $f(\phi)$ has two extreme values $f(\phi_1*)$ and $f(\phi_2*)$ where $f(\phi_1*)$ is maximal value and $f(\phi_2*)$ is the minimal value. It is easy to check that $f(\phi_1*) > f(\phi_2*)$. Therefore, in the intervals $(-\infty, \phi_1*)$ and (ϕ_2*, ∞), the function $f(\phi)$ is monotone increasing, while in the interval (ϕ_1*, ϕ_2*) the function $f(\phi)$ is monotone decreasing.

For the case $g < g^*$, , i.e., $f(\phi_1*) > f(\phi_2*) > 0$. Only one equilibrium point $E_1(\phi_1, 0)$ $(\phi_1 < 0)$ can be found. Since in the interval $(-\infty, \phi_1^*)$, the function $f(\phi)$ is monotone increasing, i.e.,$f'(\phi) > 0$ for any given ϕ in this interval. We have $J(\phi_1, 0) = -2\epsilon\phi_1 f'(\phi_1) > 0$ which means the equilibrium point is a center point.

Similar analysis can be employed for other cases. Thus we get the following result about the location and types of equilibrium points of system (10).

Proposition 2.1 *Possible equilibrium points of system* (10) *are listed below.*

1. *When $g < g^*$, there is only one equilibrium point $E_1(\phi_1, 0)$ $(\phi_1 < 0)$. This equilibrium point E_1 is a center(see Fig. 1(a)).*

2. *When $g = g^*$, there are two equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_1, 0)$ $(\phi_1 < 0 < \phi_2)$. E_1 is a center while E_2 is a cusp(see Fig. 1(b)).*

3. *When $g^* < g < 0$, there are three equilibrium points $E_i(\phi_i, 0)$ $(i = 1, 2, 3, \phi_1 < 0 < \phi_2 < \phi_2^* < \phi_3)$. Both E_1 and E_2 are center points while E_3 is a saddle point (see Fig. 1(c)).*

4. *When $g = 0$, there are two equilibrium points $E_i(\phi_i, 0)$ $(i = 1, 2, \phi_1 = 0 < \phi_2^* < \phi_2)$. E_1 is a cusp and E_2 is a saddle point (see Fig. 1(d)).*

5. *When $g > 0$, there are three equilibrium points $E_1(\phi_1, 0), E_s^{\pm}(0, Y_s^{\pm}).\phi_2^* < \phi_1$. E_1 is a saddle point, E_s^+ is an unstable node while E_s^- is stable (see Fig. 1(e)).*

From the above analysis we can see the different roles the parameter play. The integral constant g decide number of the equilibrium points while system parameters δ and ϵ determine different types of the equilibrium points.

Phase portraits of system (10) are shown in Fig. 1. Parameters are taken as $\epsilon = 1/4, c = 3, \delta = 1/2$.

3 Different types of traveling wave solutions of (5)

We will discuss the existence of Different types of traveling wave solutions of (5). We denote that $h_i = H(\phi_i, 0)$ defined by (11).

From Proposition 2.1, we see the following conclusions hold.

Proposition 3.1 *There are solitary traveling wave solutions for* (5)

1. *Suppose that $g^* < g < 0$. Then, corresponding to $H(\phi, y) = h_3$,* (5) *has a smooth solitary traveling wave solution of valley type.*

2. *Suppose that $g = 0$. Then, corresponding to $H(\phi, y) = h_2$,* (5) *has a smooth solitary traveling wave solution of valley type.*

3. *Suppose that $g = 0$. Then, corresponding to $H(\phi, y) = h, h \in (-\infty, h_2)$,* (5) *has uncountable smooth solitary traveling wave solution of bell type.*

Proof. We only prove the case 3.1 (1). Others can be treated in a similar way.

We see from (11) that there is a homogeneous orbit to the saddle point E_3 and the saddle point is located at the right side of the homogeneous orbit. By dynamical system theory [], a solitary wave solution of (5) corresponds to a homogeneousorbit of system (9). Thus, (5) has a smooth solitary traveling wave solution of valley type.

Let

$$F(\phi) = 2\,\frac{-1+\delta}{\delta+1+2\,\epsilon}\phi^3 + \frac{c}{\delta+1-\epsilon}\phi^2 - 2\,\frac{g}{\delta+1+\epsilon} + h_2\phi^{\frac{\delta+1+\epsilon}{\epsilon}}. \tag{14}$$

The homogeneous orbit to the saddle point E_3 can be expressed by

$$y = \pm F(\phi), \phi_m < \phi < \phi_2 \tag{15}$$

where $(\phi_m, 0)$ is the intercept point of the homogeneous orbit passing through the saddle E_2.

By using (16) and taking initial value $\phi(0) = \phi_m$ on a branch of the homogeneous orbit to do integration, we can have the implicit expression of the smooth solitary solution

$$\int \frac{d\phi}{\sqrt{F(\phi)}} = \pm \int d\xi \tag{16}$$

□

We note that there are two heterogeneous orbits connecting the saddle point and two node points on the $y-$axes (see the black trajectory in Fig. (1)(e), or Fig.2). Because there is a singular straight line $\phi = 0$ connecting the two nodes, according to the Fast Jumping Theory in singular traveling wave equation [10], any point near the singular straight line on the stable or instable manifold of the saddle points, $y = \phi'$ jumps in a very short time. That is, the first derivative of u changes its sign. Thus a peakon wave solution appears. There are peakon solutions for (5).

Proposition 3.2 *When $g > 0$, corresponding to $H(\phi, y) = h_2$, (5) has a peakon solution with valley type.*

Remark 3.3 *For generic δ and ϵ, the integral $\int \frac{d\phi}{\sqrt{F(\phi)}}$ cannot be expressed by elementary functions. We use numerical integration.*

The one-dimensional portrait of the smooth solitary traveling wave solution and the peakon wave solution are shown in Figs. 3 (b) and (c), respectively. Fig. 3 (a) is the profile of smooth solitary wave solutions corresponding to the family of homogeneous orbits to the right hand side of the origin.

4 Conclusions

We have found that like the CH equation, the 2-parameter family KdV-like equation also has smooth solitary wave solutions and non-smooth peakon solutions. It could be explained by the similarity with the CH equation after multiplying (5) by u. We point out that many studies would be carried on to the Kdv-like 2 parameter advection-dispersion equation (5). Under the special parameter conditions, we just have obtained the existence of different types of traveling wave solutions. For other parameter conditions, the questions of solitary wave solutions remain further researches.

Acknowledgements

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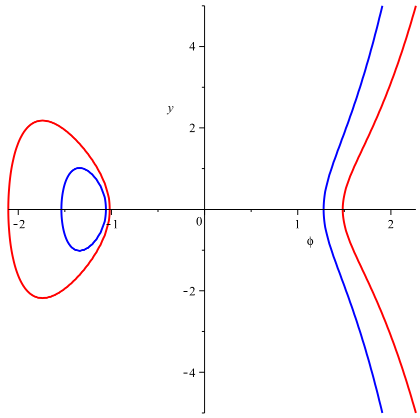
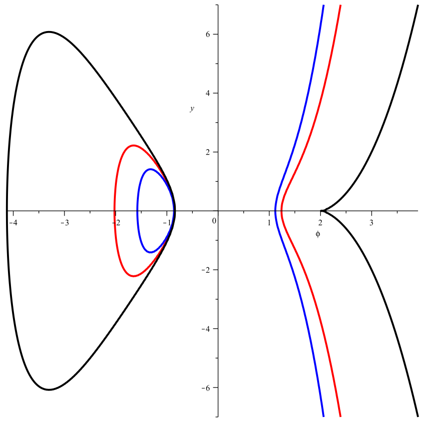
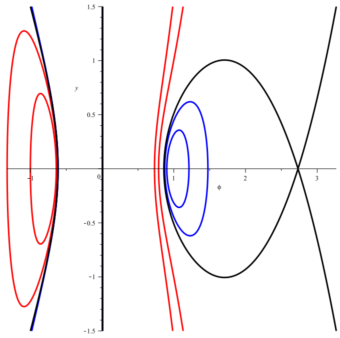


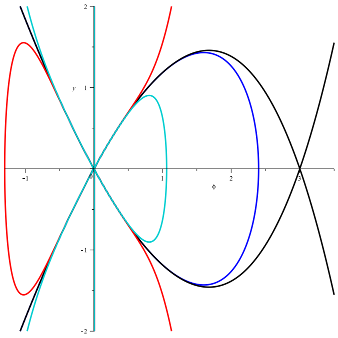
(a) $g < g^*$



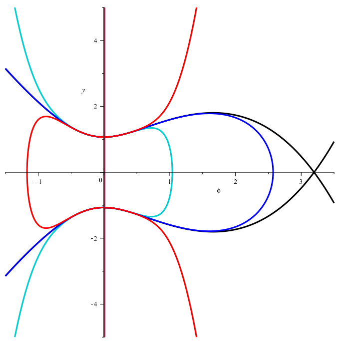
(b) $g = g^*$



(c) $g^* < g < 0$



(d) $g = 0$



(e) $g > 0$

Figure 1: Possible phase portrait for system (10)

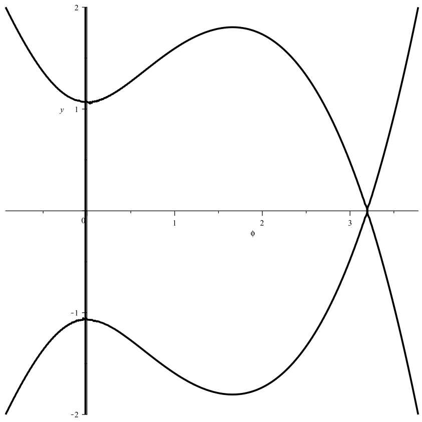
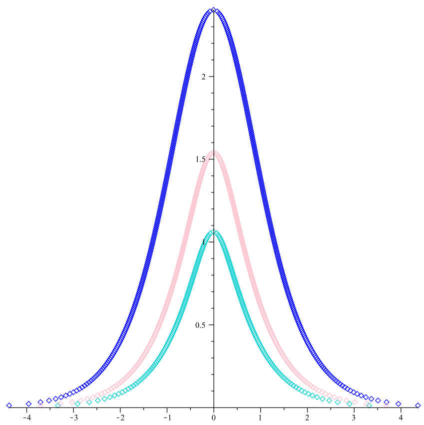
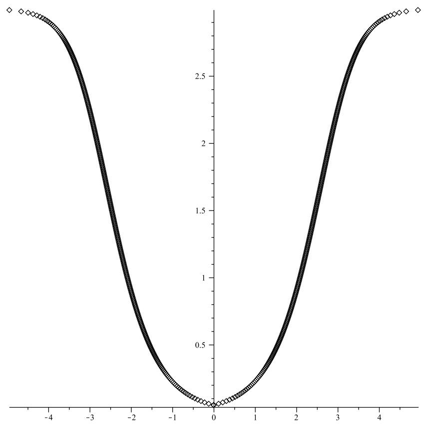


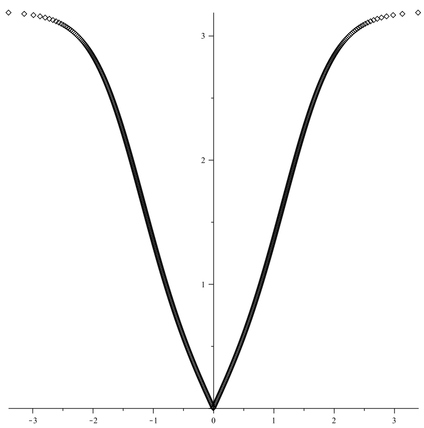
Figure 2: Orbits connecting the saddle point and the node points



(a) Solitary wave of bell type



(b) Solitary wave of valley type



(c) Peakon wave with valley type

Figure 3: Different types of solitary wave solutions of (5)