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Tropical Normal Functions - Higher Abel-Jacobi Invariants of TropicalCycles

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ABSTRACT

We consider the variation of tropical Hodge structure (TVHS) associated to families of tropical varieties. The family of the tropical intermediate Jacobians of the associated tropical Hodge structure defines a bundle of tropical Jacobians, whose sections we call the tropical normal functions. We define formal sequential derivatives of these functions on the base with respect to the natural Gauss-Manin connection as the Hodge theoretic invariants detecting tropical cycles in the fibers. The associated invariants which are defined inductively are the higher Abel-Jacobi invariants in the tropical category. They naturally identify the tropical Bloch-Beilinson filtration on the tropical Chow group. We examine this construction on the moduli of tropical curves with marked points, in order to study the tropical tautological classes in the tautological ring of trop. The expectation is the nontriviality of these cycles could be examined with less complexity in the tropical category. The construction is compatible with the tropicalization functor on the category of schemes, and the aforementioned procedure will also provide an alternative way to examine the relations in the tautological ring of g,n in the schemes category.

Keywords: Family of tropical manifolds; Tropical hodge structure; Tropical normal function

INTRODUCTION

The study of the tautological classes in the Chow ring of the moduli space of curves of genus g with n-marked points has been under investigation in the last decays in different fields of researches in mathematics and also in Physical sciences. The combinatorial framework of these objects makes them interesting for the experts in different areas. On the other hand the difficulties of doing specific calculations with the algebraic and homological cycles in the Chow ring of Mg,n have forced the experts to search for new techniques that enable them, how much more characterize the classes. This article is a text toward this purpose to provide new methodologyto study the generators and relations in the tautological ring of Mg,n [1].

The motivation we follow toward the above problem is through the tropical category. In fact the tropical manifolds and their tropical cycles can be constructed through the tropicalization functor applied to algebraic varieties. The idea is, the non-triviality of algebraic cycles in the Chow group [2-5].

of Mg,n can be checked out via this functor. In other words if the cycle has non-trivial invariants after the tropicalization procedure then the original algebraic cycle must be non-trivial. A natural expectation is tropicalizing the objects and the invariants may make the calculations simpler.

Historically Normal functions were introduced by H. Poincare. Higher cycle classes and Abel- Jacobi invariants were used by Green and Griffiths to show the non-triviality of a special zero cycle presented by Faber-Pandharipande. The cycle was originally from the pull back of a tautological cycle over the self product of the universal curve C2. They calculate the infinitesimal invariants of the normal function associated to a thickening of the cycle in a generic family. The resulting cycle is homologicaly trivial and its Abel-Jacobi image is zero. But the second infinitesimal invariant of the thickening cycle is non-zero, which shows it is rationally non-trivial.

The purpose of this text is to introduce tropical normal functions as sections of intermediate Jacobians of tropical variation of Hodge structure, TVHS. The work of this paper maybe explained in two ways. The first, one of the current research programs in tropical geometry involves possible ways to generalize geometric concepts to tropical category. In this way our definition of tropical normal function introduces a new concept in tropical geometry, as an extension the concept of normal functions in the category of complex VHS on quasi projective schemes over C into the tropical category. We also define higher normal functions and higher Abel-Jacobi maps in the tropical sense. The second, is

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that we use this concept as a tool to study algebraic cycles both in the tropical schemes in its own and also in the original category of schemes over C. This not only gives a method to investigate intersection theory and Chow groups of tropical varieties, but also a way to check out relations and non-triviality of the original cycles in the scheme category. By the way using the properties of the tropicalization functor; the relations in the category of schemes imply same relations between the corresponding tropical cycles. We apply this simple fact to study tropical tautological classes in the tropical Chow group of the moduli of tropical curves with marked points [6-10].

Let T = R −∞ be the tropical semifield. Tropical varieties are finite dimensional polyhe-dral complexes with specific affine structure. For example TPn is a smooth projective tropical variety homeomorphic to n-simplex. The restriction of the tropical structure to the interior of a k- dimensional face of TPn makes it into Rk . Projective tropical varieties are polyhedral complexes in TPn. Examples of this happens as tropical limit of an algebraic family Xt CPn, C when

|t| → ∞. The fundamental theorem of tropical geometry states that log|t|(Xt) TPn converges to an n-dimensional weighted polyhedral complex. A smooth projective tropical variety is a closed subset of TPn which on standard affine charts restricts to a smooth polyhedral complex in Tn.

[[76, 44, 82, 31, 63, 89]].s is important to prevent future cases of accidental household chlorine inhalation injuries.

where α is the character in K[M] corresponding to m M and | . |x : K[M] −→ R is the semi-norm extending | . | on K = C(t). The tropicalization is a non-archimedean moment map. [[99], [100]].The singular (co)homology of tropical varieties is defined in a natural way as for topological spaces. There is a natural generalization (definition) of Dolbeault cohomology to the tropical category (see Section 2 below). If X CPn × S is a holomorphic 1-parameter family of schemes over a disc S, with a tropical limit X∞ TPn, then one can define

that converges to Hp,q , where X(k) is the k-skeleton of X, [[59]]. Applying the tropicalization functor to a family X → S yields a family of tropical varieties defined over a base tropical variety S. The (tropical) cohomologies of the fiber varieties Xs vary continuously with s, and form a vector bundle with fibers.

OVERVIEW OF TROPICAL VARIETIES

The materials in this section are well known and can be found in various texts [[44, 76, 77, 96, 29, 41, 80, 63, 89, 4, 26, 19, 28, 37, 40, 38, 71, 84, 85]]. We work over the tropical ring Rtrop[x1, ..., xn] where Rtrop = R with tropical operations. A tropical polynomial is of the form The zero set V (f) Rn is a tropical hypersurface. We employ the two latices M = Zn and N = Hom(M, Z). Denote MR = M R and NR = N R and by , the natural pairing between them. A tropical function is a map f : MR R given by (1.1) whereS N . The Newton polytope of f is the convex hull ∆S = Conv(S) NR. Recall, an affine linear automorphism of MR is given as m Am + b, A GLn(R), b MR. A tropical manifold is a real topological manifold X with an atlas of coordinate charts i : Ui MR such that the transitionsi j 1 are affine linear automorphisms. We allow the manifolds to have boundaries and singularities.

A polyhedron σ Rn is an intersection of finite closed half spaces. We denote its boundary with ∂σ and its interior by int(σ). We call a compact polyhedron a polytope. A polyhedral decomposition ofa polyhedron ∆ NR is a set P of polyhedra in NR namely cells such that Say a collection of fan structures {Stv|v P [0]} are compatible if the induced fan structures on τ from all v τ are equivalent. A tropical manifold is a pair (X, P) where X is a tropical affine manifold with singularities obtained from polyhedral decomposition P of X and compatible collection {Stv|v P (0)} of fan structures.

that converges to Hp,q , where X(k) is the k-skeleton of X, [[59]]. Applying the tropicalization functor to a family X → S yields a familyof tropical varieties defined over a base tropical variety S. The (tropical) cohomologies of the fiber varieties Xs vary continuously with s, and form a vector bundle with fibers.

An ideal I is homogeneous if all the monomials appearing in a generator of I have the same total degree. Then I defines a variety in Pn−1 \ {xj = 0, j}. Its image under the map val is a subset of the tropical projective space TPn−1. A tropical projective variety is a subset of TPn−1 of the form where I is a homogeneous ideal of K[x±1 1, ..., x±n 1]. A tropical curve is an embedded graph in TP2 which is dual to the regular subdivision ∆ of support of a tropical polynomial f . We can construct tropical varieties via the functor called tropicalization applied to the schemes in algebraic geometry. The functor is an analogue of the moment map in toric geometry. Let X = X(∆) to be a toric variety, with ∆ is a rational polyhedral fanin NR. The tropicalization on ∆ is given by

TROPICAL HOMOLOGY AND ALGEBRAIC CYCLES

The singular homology of a tropical variety X, is defined naturallyas usual varieties [[59]]. The trop- ical Dolbeault cohomology groups are defined as the cohomology of Dolbeault complex (A•,•, dJ, dJJ) of smooth real differential forms on the analytic space Xan, where In the rest of the section we briefly introduce the notion of tropical algebraic cycles and tropical Chow group, [[77, 96, 94, 30, 6, 5, 10, 11, 80, 82, 31, 63, 4, 38]]. We first introduce the concept of the weight function on polyhedrans. Assume X is a tropical variety. Let ∆(k) be the set of all cones of codimension k on X. If σ ∆(k), τ ∆(k+1), Let Nσ be the lattice span of σ, and let nσ,τ σ be the primitive integer vector whose image generates the 1-dimensional lattice Nσ/Nτ . We denote the free abelian group on all such σ ∆(k) by Ztrop(X). A tropical regular function on a tropical cycle Z Ztrop(X) is a piece-wise affine function with integral slopes which isthe restriction of a concave function on Rn, i.e of the form where fσ : Nσ → R denotes the linear part of the affine function f |σ and codimστ = 1. We define Rk the subgroup of divisors (f) where fis a function on Z Zk. Finally the tropical Chow group of degree kis defined as CHk = Zk/Rk. The total tropical Chow ring is One may formulate the above properties in terms of the Hodge classes Hp,p instead. We explain the case of curves as the simplest case [[38, 37, 28, 4, 89, 105, 82, 80, 54, 26, 19, 20, 5, 6]]. Let C = (V, E) be a graph. If vV is a vertex its valance is the number of edges connected to v. A metric graph is a graph with length function A tropical curve is a connected metric graph C where val(v) ≥ 2 forall v V . A divisor on C is an element of the free abelian group generated by the points of |C|, i.e D = aiPi. A rational function on |C| is a continuous piece-wise linear function f : |C| → R with integer slopes. One can adjust this by specifying.

TROPICAL NORMAL FUNCTIONS AND (HIGHER) ABEL-JACOBI INVARIANTS

In this section we define tropical Hodge theoretic invariants detecting tropical cycles in the tropical Chow group of tropical varieties. The corresponding invariants in the category of quasiprojective schemes over C are already well known [[66, 67, 68, 69, 51, 52, 73, 102, 106, 107, 108, 109, 110, 55, 17]]. Assume X CPn × S is a holomorphic 1-parameter family of schemes over a discS, with a tropical limit X∞ TPn, then one can define We consider a family of projective tropical varieties defined over a quasi-projective tropical variety S. Because the discussion is local such a family can be simply be a product X × S. The family can be defined by an affine map with parameters varying in a tropical variety. The fibervarieties Xs vary continuously with s, and their cohomologies form a vector bundle with fibers. The above two pieces can be collapsed to define a filtration on tropical Chow group of X, where its i-th graded piece defines the i-th cycle class map on the tropical cycles,which we denote by trop a shift according to the second grading. We record this issue in a split short exact sequence of Abelian groups which maps these cycles as

The aforementioned construction is justified by the tropicalization functor. The reader may know that the construction is well established in the category of schemes, [63]. If we apply the tropicalization functor to a family of projective varieties over a quasi-projective base X → S we obtain a family of tropical varieties trop(X)→ trop(S) where all the construction applies to. In fact the definition shows that the invariants just defined are the tropicalization of the corresponding invariant in the category of projective schemes. An example we may identify the BB-filtration mentioned above for a product of tropical curves.

TAUTOLOGICAL CLASSES IN MODULI OF TROPICAL CURVES WITH MARKED POINTS

We apply the technology of the previous section to tautological classes in the tropical Chow ring of Mg,n , the moduli of tropical curves of genus g with n marked points, [[78, 60, 79, 44, 30, 19, 21, 24, 42, 81, 90, 99, 100]]. The idea is a family of tropical curves over a base can also be studied over the moduli of tropical curves, as a universal curve. Lets recall the definition of a tropical curve with marked points. We always assume a graphs Γ = (V, E), is weighted, i.e. come with a weight function. The above decomposition is the base of some analysis of cycle classes in the Chow ring of Cn .

It is more convenient to consider the tautological subrings of tropical cycles or their corresponding Hodge classes generated by the diagonals Δtrop of two marking points and the tropical classes ψtrop bundles at the marked points, and the Morita-Mumford-Miller classes κd. According to the major result in [[87]] a Schur-Weyl duality argument applies with an application of the theorem of Ancona, [[8]] to obtain.

CONCLUSION

The study of the tautological classes in the Chow ring of the moduli space of curves of genus g with n-marked points has been under investigation in the last decays in different fields of researches in mathematics and also in Physical sciences. The combinatorial framework of these objects makes them interesting for the experts in different areas. On the other hand the difficulties of doing specific calculations with the algebraic and homological cycles in the Chow ring of Mg,n have forced the experts to search for new techniques that enable them, how much more characterize the classes. This article is a text toward this purpose to provide new methodology to study the generators and relations in the tautological ring of Mg,n.

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CONTRIBUTIONS

The normal functions associated to variation of tropical Hodge structure has been introduced together with the higher tropical cycle classes and Higher Abel-Jacobi Invariants of tropical cycles in the fibered variety. We have used these invariants to study generators and relations in the tropical tautological ring of the moduli or tropical curves of genus g and n-marked points.

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