Mathematica Aeterna, Vol. 4, 2014, no. 2, 119 - 122

Three inequalities for the incomplete polygamma function

Banyat Sroysang

Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University, Pathumthani 12121 Thailand
banyat@mathstat.sci.tu.ac.th

Abstract

In this paper, we present three inequalities for the incomplete polygamma function.

Mathematics Subject Classification: 26D15

Keywords: Polygamma function, inequality

1 Introduction

The polygamma function of order $n \in \mathbb{N}$ is defined by

$$\Psi_n(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x),$$

where Γ is the gamma function and $x > 0$.

The polygamma function may be represented as

$$\Psi_n(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt,$$

where $n \in \mathbb{N}$ and $x > 0$.

The incomplete polygamma function of order $n \in \mathbb{N}$ is defined by

$$\Psi_n(a, b, x) = (-1)^{n+1} \int_a^b \frac{t^n e^{-xt}}{1 - e^{-t}} dt,$$

where $a, b, x > 0$.

In 2006, Laforgia and Natalini [1] showed that

$$\Psi_m(x)\Psi_n(x) \geq \Psi^2_{\frac{m+n}{2}}(x)$$

where $m, n, \frac{m+n}{2} \in \mathbb{N}$ and $x > 0$.

In 2011, Sulaiman [3] gave the inequalities as follows.

$$\Psi_m^{1/p}(x)\Psi_n^{1/q}(x) \geq \Psi_{\frac{m}{p}+\frac{n}{q}}(x) \tag{1}$$

where $m, n, \frac{m}{p} + \frac{n}{q} \in \mathbb{N}$, $x > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\Psi_n^{1/p}(px)\Psi_n^{1/q}(qy) \geq \Psi_n\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \tag{2}$$

where n is a positive odd integer, $x, y > 1$, $\frac{1}{x} + \frac{1}{y} \leq 1$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$-\Psi_n^{1/p}(px)\Psi_n^{1/q}(qy) \leq \Psi_n\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \tag{3}$$

where n is a positive even integer, $x, y > 1$, $\frac{1}{x} + \frac{1}{y} \leq 1$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In 2013, Sroysang [2] presented the generalizations for the inequalities (1), (2) and (3).

In this paper, we present three inequalities for the incomplete polygamma function similar to the inequalities (1), (2) and (3).

2 Results

Theorem 2.1. *Let $m, n \in \mathbb{N}$, $a, b, x > 0$ and $p, q > 1$ be such that $\frac{m}{p} + \frac{n}{q} \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Psi_m^{1/p}(a,b,x)\Psi_n^{1/q}(a,b,x) \geq \Psi_{\frac{m}{p}+\frac{n}{q}}(a,b,x).$$

Proof. By the generalized Hölder inequality,

$$\begin{aligned}
\Psi_{\frac{m}{p}+\frac{n}{q}}(a,b,x) &= (-1)^{\frac{m}{p}+\frac{n}{q}+1}\int_a^b \frac{t^{\frac{m}{p}+\frac{n}{q}}e^{-xt}}{1-e^{-t}}dt \\
&= (-1)^{\frac{m+1}{p}+\frac{n+1}{q}}\int_a^b \left(\frac{t^{\frac{m}{p}}e^{-\frac{x}{p}t}}{(1-e^{-t})^{1/p}}\right)\left(\frac{t^{\frac{n}{q}}e^{-\frac{x}{q}t}}{(1-e^{-t})^{1/q}}\right)dt \\
&\leq (-1)^{\frac{m+1}{p}+\frac{n+1}{q}}\left(\int_a^b \frac{t^m e^{-xt}}{1-e^{-t}}dt\right)^{1/p}\left(\int_a^b \frac{t^n e^{-xt}}{1-e^{-t}}dt\right)^{1/q} \\
&= \Psi_m^{1/p}(a,b,x)\Psi_n^{1/q}(a,b,x).
\end{aligned}$$

□

Corollary 2.2. *Let $m, n \in \mathbb{N}$ and $a, b, x > 0$. Then*

$$\Psi_m(a,b,x)\Psi_n(a,b,x) \geq \Psi^2_{\frac{m+n}{2}}(a,b,x).$$

Theorem 2.3. *Let n be a positive odd integer, and let $x, y > 1$ and $p, q > 1$ be such that $x + y \leq xy$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Psi_n^{1/p}(a,b,px)\Psi_n^{1/q}(a,b,qy) \geq \Psi_n\left(a,b,\frac{x^p}{p}+\frac{y^q}{q}\right).$$

Proof. Note that $xy \leq \dfrac{x^p}{p} + \dfrac{y^q}{q}$. By the generalized Hölder inequality,

$$\begin{aligned}
\Psi_n\left(a,b,\frac{x^p}{p}+\frac{y^q}{q}\right) &= \int_a^b \frac{t^n e^{-t\left(\frac{x^p}{p}+\frac{y^q}{q}\right)}}{1-e^{-t}}dt \\
&\leq \int_a^b \frac{t^n e^{-txy}}{1-e^{-t}}dt \\
&\leq \int_a^b \frac{t^n e^{-t(x+y)}}{1-e^{-t}}dt \\
&= \int_a^b \left(\frac{t^{\frac{n}{p}} e^{-xt}}{(1-e^{-t})^{1/p}}\right)\left(\frac{t^{\frac{n}{q}} e^{-yt}}{(1-e^{-t})^{1/q}}\right)dt \\
&\leq \left(\int_a^b \frac{t^n e^{-pxt}}{1-e^{-t}}dt\right)^{1/p}\left(\int_a^b \frac{t^n e^{-qyt}}{1-e^{-t}}dt\right)^{1/q} \\
&= \Psi_n^{1/p}(a,b,px)\Psi_n^{1/q}(a,b,qy).
\end{aligned}$$

□

Corollary 2.4. *Let n be a positive odd integer, and let $x, y > 1$ be such that $x + y \leq xy$. Then*

$$\Psi_n(a,b,2x)\Psi_n(a,b,2y) \geq \Psi_n^2\left(a,b,\frac{x^2+y^2}{2}\right).$$

Theorem 2.5. *Let n be a positive even integer, and let $x, y > 1$ and $p, q > 1$ be such that $x + y \leq xy$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$-\Psi_n^{1/p}(a,b,px)\Psi_n^{1/q}(a,b,qy) \leq \Psi_n\left(a,b,\frac{x^p}{p}+\frac{y^q}{q}\right).$$

Proof. Note that $xy \le \frac{x^p}{p} + \frac{y^q}{q}$. By the generalized Hölder inequality,

$$\begin{aligned}
\Psi_n\left(a, b, \frac{x^p}{p} + \frac{y^q}{q}\right) &= -\int_a^b \frac{t^n e^{-t\left(\frac{x^p}{p}+\frac{y^q}{q}\right)}}{1-e^{-t}} dt \\
&\ge -\int_a^b \frac{t^n e^{-txy}}{1-e^{-t}} dt \\
&\ge -\int_a^b \frac{t^n e^{-t(x+y)}}{1-e^{-t}} dt \\
&= -\int_a^b \left(\frac{t^{\frac{n}{p}} e^{-xt}}{(1-e^{-t})^{1/p}}\right)\left(\frac{t^{\frac{n}{q}} e^{-yt}}{(1-e^{-t})^{1/q}}\right) dt \\
&\ge -\left(\int_a^b \frac{t^n e^{-pxt}}{1-e^{-t}} dt\right)^{1/p} \left(\int_a^b \frac{t^n e^{-qyt}}{1-e^{-t}} dt\right)^{1/q} \\
&= -\Psi_n^{1/p}(a, b, px)\Psi_n^{1/q}(a, b, qy).
\end{aligned}$$

□

References

[1] A. Laforgia and P. Natalini, Turan-type inequalites for some special functions, J. Ineq. Pure Appl. Math., 2006, **7**(1), Art. 32.

[2] B. Sroysang, Inequalities for the polygamma function, Math. Aeterna, 2013, **3**(4), 249–252.

[3] W. T. Sulaiman, Turan inequalites for the digamma and polygamma functions, South Asian J. Math., 2011, **1**(2), 49–55.

Received: January, 2014