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The properties of biderivations on Heisenberg superalgebras

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Abstract

Let H be a Heisenberg superalgebra. In this paper, the definition of biderivations and the properties of biderivations on Lie superalgebras are introduced. And some properties of biderivations on Heisenberg superalgebras are introduced by the definition of Heisenberg superalgebras.

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1 Introduction

Recently,some researchers were interested in biderivations of Lie algebras [1] and the definition of Heisenberg superalgebras [2].The aim of this paper is to introduce the properties of biderivations on Heisenberg superalgebras.

2 Preliminary Notes

Definition 2.1 *Let* $L = L_{\overline{0}} + L_{\overline{1}}$ *be a superalgebra whose multiplication is denoted by a pointed bracket* $[-,-]$. *This implies in particular that* $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ *for all* $\alpha, \beta \in \mathbb{Z}_2$. *We call* L *is a Lie superalgebra if the multiplication satisfies the following identities,*

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

$$[x,[y,z]]+(-1)^{|x||y|}[y,[z,x]]+(-1)^{|y||z|}[z,[x,y]]=0$$

for all $x \in L_{|x|}$, $y \in L_{|y|}$, $z \in L_{|z|}$ *and* $| x |, | y |, | z | \in \mathbb{Z}_2$.

Definition 2.2 *Let* L *be a Lie algebra. A bilinear map* $\varphi : L \times L \longrightarrow L$ *is called a biderivation of* L *if it is a derivation with respect to both components, meaning that*

$$\varphi([x,y],z) = [x,\varphi(y,z)] + [\varphi(x,z),y]$$

and

$$\varphi(x,[y,z]) = [\varphi(x,y),z] + [y,\varphi(x,z)]$$

for all $x,y,z \in L$.

Definition 2.3 *Let* H *be a Lie superalgebra.* H *is called a Heisenberg superalgebra if* $[H,H] = C(H)$ *and* $\dim C(H) = 1$.

Remark 2.4 *According to the definition of Heisenberg superalgebras, we have* $\dim C(H) = 1$ *and* $C(H) = c$*, which* $c \in H_{\bar{0}}$ *and* $c \neq 0$*. Owing to* $[H,H] = C(H) \subseteq H_{\bar{0}}$*, there exists a grade anti-symmetic bilinear function* ψ*, s.t.* $[x,y] = \psi(x,y)c$.

Definition 2.5 *Let* L *be a Lie superalgebra. A bilinear map* $\varphi : L\times L \longrightarrow L$ *is called a biderivation of* L *if it is a derivation with respect to both components,meaning that*

$$\varphi([x,y],z) = [x,\varphi(y,z)] + (-1)^{d(y)(d(\varphi)+d(z))}[\varphi(x,z),y]$$

and

$$\varphi(x,[y,z]) = [\varphi(x,y),z] + (-1)^{d(y)(d(\varphi)+d(x))}[y,\varphi(x,z)]$$

for all $x,y,z \in L$.

3 Main Results

Theorem 3.1 *Let* φ *be a biderivation on Lie superalgebras. Then*

$$[\varphi(x,y),[u,v]] = (-1)^{(d(y)+d(u))d(\varphi)}[[x,y],\varphi(u,v)]$$

for all $x,y,u,v \in L$.

Proof. Since φ is a biderivation on Lie superalgebras,then we have

$$\varphi([x,y],z) = [x,\varphi(y,z)] + (-1)^{d(y)(d(\varphi)+d(z))}[\varphi(x,z),y]$$

and

$$\varphi(x,[y,z]) = [\varphi(x,y),z] + (-1)^{d(y)(d(\varphi)+d(x))}[y,\varphi(x,z)]$$

for all $x, y.z \in L$

We compute $\varphi([x,u],[y,v])$ in two different ways. On one hand, since φ is a biderivation in the first formula. We have that

$$\varphi([x,u],[y,v]) = [x, \varphi(u,[y,v])] + (-1)^{d(u)(d(\varphi)+d(y)+d(v))}[\varphi(x,[y,v]),u].$$

Using the fact that φ is a biderivation in the second formula, we further have that

$$\begin{gathered}\varphi([x,u],[y,v]) = [x,[\varphi(u,y),v]]+\\ (-1)^{d(y)(d(\varphi)+d(u))}[x,[y,\varphi(u,v)]]+\\ (-1)^{d(u)(d(\varphi)+d(y)+d(v))}[[\varphi(x,y),v],u]+\\ (-1)^{d(u)(d(\varphi)+d(y)+d(v))+d(y)(d(\varphi)+d(x))}[[y,\varphi(x,v)],u].\end{gathered}$$

On the other hand, computing $\varphi([x,u],[y,v])$ in a different way we have that

$$\begin{gathered}\varphi([x,u],[y,v]) = [[x,\varphi(u,y)],v]+\\ (-1)^{d(u)(d(\varphi)+d(y))}[[\varphi(x,y),u],v]+\\ (-1)^{d(y)(d(\varphi)+d(x)+d(u))}[y,[x,\varphi(u,v)]]+\\ (-1)^{d(y)(d(\varphi)+d(x)+d(u))+d(u)(d(\varphi)+d(v))}[y,[\varphi(x,v),u]].\end{gathered}$$

By comparing the two equalities we have that

$$[\varphi(x,y),[u,v]] = (-1)^{(d(y)+d(u))d(\varphi)}[[x,y],\varphi(u,v)]$$

for all $x, y, u, v \in L$.

$\square$

Theorem 3.2 *Let $C = \{\lambda c | \lambda \in \mathbb{C}\}$. We have C is the center of H, there exists $[x,y] \in C$ for all $x, y \in H$ and $z = \eta[x,y], \eta \in \mathbb{C}, \exists x, y \in L$, for $z \in C$.*

Proof. Let $H = H_{\bar{0}} + H_{\bar{1}}$ be a Heisenberg superalgebra.

Case 1 $\dim H$ is odd.

Since $H = H_{\bar{0}} + H_{\bar{1}}$ and $H_{\bar{0}}$ are Heisenberg algebra whose dimension is odd, so $\dim H_{\bar{1}}$ is even. Then there is a group of basis $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n, c\}$ of $H_{\bar{0}}$, analogously, the basis of $H_{\bar{1}}$ are $\{s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n\}$. Then the basis of $H = H_{\bar{0}} + H_{\bar{1}}$ are $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n, c, s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n\}$.

Let

$$x = \lambda_1 c + \sum_{i=1}^{n} \lambda_{i+1} e_i + \sum_{i=1}^{n} \lambda_{i+n+1} f_i + \sum_{i=1}^{n} \eta_i s_i + \sum_{i=1}^{n} \eta_{i+n} t_i$$

$$y = \xi_1 c + \sum_{i=1}^{n} \xi_{i+1} e_i + \sum_{i=1}^{n} \xi_{i+n+1} f_i + \sum_{i=1}^{n} \zeta_i s_i + \sum_{i=1}^{n} \zeta_{i+n} t_i$$

$\lambda_i, \eta_i, \xi_i, \zeta_i (i = 1, 2, \ldots, 2n+1)$

$$[x, y] = [\lambda_1 c + \sum_{i=1}^{n} \lambda_{i+1} e_i + \sum_{i=1}^{n} \lambda_{i+n+1} f_i + \sum_{i=1}^{n} \eta_i s_i + \sum_{i=1}^{n} \eta_{i+n} t_i,$$

$$\xi_1 c + \sum_{i=1}^{n} \xi_{i+1} e_i + \sum_{i=1}^{n} \xi_{i+n+1} f_i + \sum_{i=1}^{n} \zeta_i s_i + \sum_{i=1}^{n} \zeta_{i+n} t_i]$$

$$= [\sum_{i=1}^{n} \lambda_{i+1} e_i, \sum_{i=1}^{n} \xi_{i+n+1} f_i] + [\sum_{i=1}^{n} \lambda_{i+n+1} f_i, \sum_{i=1}^{n} \xi_{i+1} e_i]$$

$$+[\sum_{i=1}^{n} \eta_i s_i, \sum_{i=1}^{n} \zeta_{i+n} t_i] + [\sum_{i=1}^{n} \eta_{i+n} t_i, \sum_{i=1}^{n} \zeta_i s_i]$$

$$= \sum_{i=1}^{n} (\lambda_{i+1} + \xi_{i+n+1}) c - \sum_{i=1}^{n} (\lambda_{i+n+1} + \xi_{i+1}) c$$

$$+ \sum_{i=1}^{n} (\eta_i + \zeta_{i+n}) c + \sum_{i=1}^{n} (\eta_{i+n} + \zeta_i) c = \lambda c$$

So,we have $[x, y] = \lambda c, \lambda \in \mathbb{C}$,i.e.$[x, y] \in C$.

There exists $[x, y] = \eta_1 c, \eta_1 \neq 0, \exists x, y \in H$ and $z = \eta_2 c, \exists \eta_2 \in \mathbb{C}$ for all $z \in C$, so we have $\eta[x, y] = \eta\eta_1 c$ and let $\eta = \frac{\eta_2}{\eta_1}$, we have $z = \eta[x, y]$.

Case 2 $\dim H$ is even.

Since $H = H_{\bar{0}} + H_{\bar{1}}$ is a Heisenberg superalgebra whose dimension is even and $H_{\bar{0}}$ is Heisenberg algebra whose dimension is odd. So $\dim H_{\bar{1}}$ is odd. Then there is a group of basis $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n, c\}$ of $H_{\bar{0}}$, analogously, the basis of $H_{\bar{1}}$ are $\{s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n, \omega\}$. Then the basis of $H = H_{\bar{0}} + H_{\bar{1}}$ are $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n, c, s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n, \omega\}$, which $\psi(\omega, \omega) = 1$.

Let

$$x = \lambda_1 c + \sum_{i=1}^{n} \lambda_{i+1} e_i + \sum_{i=1}^{n} \lambda_{i+n+1} f_i + \eta_1 \omega + \sum_{i=1}^{n} \eta_{i+1} s_i + \sum_{i=1}^{n} \eta_{i+n+1} t_i$$

$$y = \xi_1 c + \sum_{i=1}^{n} \xi_{i+1} e_i + \sum_{i=1}^{n} \xi_{i+n+1} f_i + \zeta_1 \omega + \sum_{i=1}^{n} \zeta_{i+1} s_i + \sum_{i=1}^{n} \zeta_{i+n+1} t_i$$

$\lambda_i, \eta_i, \xi_i, \zeta_i (i = 1, 2, \ldots, 2n+1)$

$$[x, y] = [\lambda_1 c + \sum_{i=1}^{n} \lambda_{i+1} e_i + \sum_{i=1}^{n} \lambda_{i+n+1} f_i + \eta_1 \omega + \sum_{i=1}^{n} \eta_{i+1} s_i + \sum_{i=1}^{n} \eta_{i+n+1} t_i,$$

$$\xi_1 c+\sum_{i=1}^{n}\xi_{i+1}e_i+\sum_{i=1}^{n}\xi_{i+n+1}f_i+\zeta_1\omega+\sum_{i=1}^{n}\zeta_{i+1}s_i+\sum_{i=1}^{n}\zeta_{i+n+1}t_i]$$

$$=[\sum_{i=1}^{n}\lambda_{i+1}e_i,\sum_{i=1}^{n}\xi_{i+n+1}f_i]+[\sum_{i=1}^{n}\lambda_{i+n+1}f_i,\sum_{i=1}^{n}\xi_{i+1}e_i]$$

$$+\eta_1\zeta_1[\omega,\omega]+[\sum_{i=1}^{n}\eta_{i+1}s_i,\sum_{i=1}^{n}\zeta_{i+n+1}t_i]+[\sum_{i=1}^{n}\eta_{i+n+1}t_i,\sum_{i=1}^{n}\zeta_{i+1}s_i]$$

$$=\sum_{i=1}^{n}(\lambda_{i+1}+\xi_{i+n+1})c-\sum_{i=1}^{n}(\lambda_{i+n+1}+\xi_{i+1})c+\eta_1\zeta_1[\omega,\omega]$$

$$+\sum_{i=1}^{n}(\eta_{i+1}+\zeta_{i+n+1})c+\sum_{i=1}^{n}(\eta_{i+n+1}+\zeta_{i+1})c=\lambda c$$

So,we have $[x,y]=\lambda c,\lambda\in\mathbb{C}$, i.e., $[x,y]\in C$.

There exists $[x,y]=\eta_1 c,\eta_1\neq 0,\exists x,y\in H$ and $z=\eta_2 c,\exists\eta_2\in\mathbb{C}$ for all $z\in C$, so we have $\eta[x,y]=\eta\eta_1 c$ and let $\eta=\frac{\eta_2}{\eta_1}$, we have $z=\eta[x,y]$. □

Theorem 3.3 *The properties of biderivations on Heisenberg superalgebras.*

$$(1)\varphi(0,z)=0,\varphi(x,0)=0,\forall x,z\in H.$$

$$(2)\varphi(c,z)=\lambda c,\varphi(x,c)=\eta c,\forall x,z\in H.$$

$$(3)\varphi(c,c)=0.$$

$$(4)\varphi(x,c)=-(-1)^{d(\varphi)d(x)}\varphi(c,x),\forall x\in H.$$

Proof. 1. According to

$$\varphi([x,y],z)=[x,\varphi(y,z)]+(-1)^{d(y)(d(\varphi)+d(z))}[\varphi(x,z),y]$$

and

$$\varphi(x,[y,z])=[\varphi(x,y),z]+(-1)^{d(y)(d(\varphi)+d(x))}[y,\varphi(x,z)]$$

for all $x,y,z\in L$.

Let $x=y=c$ in the first formula, then we have

$$\varphi([c,c],z)=[c,\varphi(c,z)]+(-1)^{d(c)(d(\varphi)+d(z))}[\varphi(c,z),c].$$

Hence $\varphi(0,z)=0,\forall z\in H$.

Let $y=z=c$ in the second formula, then we have

$$\varphi(x,[c,c])=[\varphi(x,c),c]+(-1)^{d(c)(d(\varphi)+d(x))}[c,\varphi(x,c)].$$

Hence $\varphi(x,0)=0,\forall x\in H$.

2. Let $x = c$ in the first formula, then we have

$$\varphi([c, y], z) = [c, \varphi(y, z)] + (-1)^{d(y)(d(\varphi)+d(z))}[\varphi(c, z), y].$$

So we have $[\varphi(c, z), y] = 0$.According to Theorem 2.2 ,we have $\varphi(c, z) = \lambda c \in C, \forall z \in H$.

Let $y = c$ in the second formula, then we have

$$\varphi(x, [c, z]) = [\varphi(x, c), z] + (-1)^{d(c)(d(\varphi)+d(x))}[c, \varphi(x, z)].$$

Hence $\varphi(x, c) = \eta c \in C, \forall x \in H$.

3. Let $x = c$ in the second formula. According to Theorem 2.2, suppose that $[y, z] = \lambda_1 c$ for all $y, z \in L$, then we have

$$\varphi(c, \lambda_1 c) = [\varphi(c, y), z] + (-1)^{d(y)(d(\varphi)+d(c))}[y, \varphi(c, z)].$$

Hence $\varphi(c, \lambda_1 c) = 0$, i.e., $\varphi(c, c) = 0$.

4.Suppose that

$$[x, y] = 0, [x, z] = 0, [y, z] = \lambda_2 c, \lambda_2 \neq 0.$$

So we have

$$\begin{cases} \varphi(x, \lambda_2 c) = [\varphi(x, y), z] + (-1)^{d(y)(d(\varphi)+d(x))}[y, \varphi(x, z)], \\ 0 = [\varphi(y, x), z] + (-1)^{d(x)(d(\varphi)+d(y))}[x, \varphi(y, z)], \\ 0 = [\varphi(z, x), y] + (-1)^{d(x)(d(\varphi)+d(z))}[x, \varphi(z, y)], \\ 0 = [x, \varphi(y, z)] + (-1)^{d(y)(d(\varphi)+d(z))}[\varphi(x, z), y], \\ 0 = [x, \varphi(z, y)] + (-1)^{d(z)(d(\varphi)+d(y))}[\varphi(x, y), z], \\ \varphi(\lambda_2 c, x) = [y, \varphi(z, x)] + (-1)^{d(z)(d(\varphi)+d(x))}[\varphi(y, x), z]. \end{cases}$$

Let

$$[\varphi(x, y), z] = x_1; [y, \varphi(x, z)] = x_2; [\varphi(y, x), z] = x_3;$$
$$[x, \varphi(y, z)] = x_4; [\varphi(z, x), y] = x_5; [x, \varphi(z, y)] = x_6.$$

so we have

$$\begin{cases} \varphi(x, \lambda_2 c) = x_1 + (-1)^{d(y)(d(\varphi)+d(x))}x_2, \\ 0 = x_3 + (-1)^{d(x)(d(\varphi)+d(y))}x_4, \\ 0 = x_5 + (-1)^{d(x)(d(\varphi)+d(z))}x_6, \\ 0 = x_4 - (-1)^{d(x)d(y)}x_2, \\ 0 = x_6 + (-1)^{d(z)(d(\varphi)+d(y))}x_1, \\ \varphi(\lambda_2 c, x) = -(-1)^{d(y)(d(\varphi)+d(z)+d(x))}x_5 + (-1)^{d(z)(d(\varphi)+d(x))}x_3. \end{cases}$$

Hence,

$$\varphi(x, c) = -(-1)^{d(\varphi)d(x)}\varphi(c, x), \forall x \in H.$$

□

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