Mathematica Aeterna, Vol. 3, 2013, no. 7, 509 - 517

The Convolution and Neutrix Convolution Involving the Function $x^s \ln(1 + x_+)$

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Abstract

The neutrix convolutions $(1 + x)^s \ln(1 + x_+) \circledast x^r$ *and* $x^s \ln(1 + x_+) \circledast x^r$ *are evaluated for* $r, s = 0, 1, 2, \ldots$. *Further results are also given.*

AMS Mathematics Subject Classification (2000): 33B15, 33B20, 46F10.

Keywords: Convolution, neutrix convolution, neutrix limit.

1 Introduction

In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see for example Gel'fand and Shilov [5].

Definition 1.1 *Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:*

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the convolution $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle g(x), \langle f(t), \varphi(x + t) \rangle \rangle$$

for arbitrary test function φ in $\mathcal{D}$.

The classical definition of the convolution is as follows:

Definition 1.2 *If f and g are locally summable functions then the convolution $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

for all x for which the integrals exist.

Note that if f and g are locally summable functions satisfying either of the conditions (a) or (b) in Definition 1.1, then Definition 1.1 is in agreement with Definition 1.2.

Definition 1.1 is rather restrictive and so a neutrix convolution was introduced in [2]. In order to define the neutrix convolution, we first of all let τ be the function in $\mathcal{D}$, see Jones [6], satisfying the following conditions:
(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1, |x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0, |x| \ge 1$.
The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

Definition 1.3 *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution $f \circledast g$ is defined to be the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that*

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero as n tends to infinity.

Note that the convolution $f_n * g$ in this definition is in the sense of Definition 1.2, the support of f_n being bounded. Note also that the neutrix convolution in this definition, is in general non-commutative.

It was proved in [2] that if the convolution $f * g$ exists by Definition 1.1, then the neutrix convolution $f \circledast g$ exists and

$$f * g = f \circledast g,$$

showing that Definition 1.3 is a generalization of Definition 1.1.

2 Main Results

We now prove

Theorem 2.1

$$(1+x)^s \ln(1+x_+) * x_+{}^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \bigg\{ \frac{(1+x)^{r+s+1} \ln(1+x_+)}{r+s-i+1} - \frac{[H(x)+x_+]^{r+s+1} - [H(x)+x_+]^i}{(r+s-i+1)^2} \bigg\} \tag{1}$$

for $r, s = 0, 1, 2, \ldots,$ where $H(x)$ denotes Heaviside's function.

Proof. When $x < 0$, it is clear that

$$(1+x)^s \ln(1+x_+) * x_+{}^r = 0. \tag{2}$$

When $x > 0$, we have on putting $u = 1 + t$

$$\begin{aligned}(1+x)^s \ln(1+x_+) * x_+{}^r &= \int_0^x (1+t)^s \ln(1+t)(x-t)^r \, dt \\ &= \int_1^{1+x} u^s \ln u (1+x-u)^r \, du \\ &= \sum_{i=0}^{r} \binom{r}{i} (1+x)^i (-1)^{r-i} \int_1^{1+x} u^{r+s-i} \ln u \, du \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \bigg\{ \frac{(1+x)^{r+s+1} \ln(1+x)}{r+s-i+1} - \frac{(1+x)^{r+s+1} - (1+x)^i}{(r+s-i+1)^2} \bigg\}\end{aligned} \tag{3}$$

and equation (1) follows from equations (2) and (3).

Corollary 2.2

$$(1-x)^s \ln(1+x_-) * x_-{}^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \bigg\{ \frac{(1-x)^{r+s+1} \ln(1+x_-)}{r+s-i+1} - \frac{[H(-x)+x_-]^{r+s+1} - [H(-x)+x_-]^i}{(r+s-i+1)^2} \bigg\} \tag{4}$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (4) follows from equation (1) on replacing x by $-x$.

Theorem 2.3

$$
\begin{aligned}
x^s \ln(1+x_+) * x_+{}^r &= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j+r-i} \Big[\frac{(1+x)^{r+j+1} \ln(1+x_+)}{r+j-i+1} \\
&\qquad - \frac{[H(x)+x_+]^{r+j+1} - [H(x)+x_+]^i}{(r+j-i+1)^2} \Big]
\end{aligned} \tag{5}
$$

for $r, s = 0, 1, 2, \ldots$.

Proof. When $x < 0$, it is clear that

$$
x^s \ln(1+x_+) * x_+{}^r = 0. \tag{6}
$$

When $x > 0$, we have on putting $u = 1 + t$

$$
\begin{aligned}
x^s \ln(1+x_+) * x_+{}^r &= \int_0^x t^s \ln(1+t)(x-t)^r \, dt \\
&= \int_1^{x+1} (u-1)^s \ln u (1+x-u)^r \, du \\
&= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j+r-i} (1+x)^i \int_1^{1+x} u^{r+j-i} \ln u \, du \\
&= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j+r-i} \Big[\frac{(1+x)^{r+j+1} \ln(1+x)}{r+j-i+1} \\
&\qquad - \frac{(1+x)^{r+j+1} - (1+x)^i}{(r+j-i+1)^2} \Big]
\end{aligned} \tag{7}
$$

and equation (5) follows from equations (6) and (7).

Corollary 2.4

$$
\begin{aligned}
x^s \ln(1+x_-) * x_-{}^r &= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{r+s-j-i} \Big\{ \frac{(1-x)^{r+j+1} \ln(1+x_-)}{r+j-i+1} \\
&\qquad - \frac{[H(-x)+x_-]^{r+j+1} - [H(-x)+x_-]^i}{(r+j-i+1)^2} \Big\}
\end{aligned} \tag{8}
$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (8) follows from equation (5) on replacing x by $-x$.

Theorem 2.5 *The neutrix convolution* $(1+x)^s \ln(1+x_+) \circledast x^r$ *exists and*

$$
(1+x)^s \ln(1+x_+) \circledast x^r = \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1+x)^i (-1)^{r+i+k}}{k(r+s-i+1)}, \tag{9}
$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Putting $[(1+x)^s \ln(1+x_+)]_n = (1+x)^s \ln(1+x_+)\tau_n(x)$ and $u = 1+t$, we have

$$\begin{aligned}
[(1+x)^s \ln(1+x_+)]_n \circledast x^r &= \int_0^n (1+t)^s \ln(1+t)(x-t)^r\,dt \\
&\quad + \int_n^{n+n^{-n}} (1+t)^s \ln(1+t)(x-t)^r \tau_n(t)\,dt \\
&= \int_1^{n+1} u^s \ln u (1+x-u)^r\,dt \\
&\quad + \int_n^{n+n^{-n}} (1+t)^s \ln(1+t)(x-t)^r \tau_n(t)\,dt \\
&= I_1 + I_2, \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=0}^{r} \binom{r}{i} (1+x)^i (-1)^{r-i} \int_1^{n+1} u^{r+s-i} \ln u\,du \\
&= \sum_{i=0}^{r} \binom{r}{i} (1+x)^i (-1)^{r-i} \Big\{ \frac{(n+1)^{r+s-i+1}\ln(n+1)}{r+s-i+1} \\
&\quad - \frac{(1+n)^{r+s-i+1} - 1}{(r+s-i+1)^2} \Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} I_1 &= \operatorname*{N-lim}_{n\to\infty} \sum_{i=0}^{r} \binom{r}{i} \frac{(1+x)^i(-1)^{r-i}}{r+s-i+1} \\
&\quad \times \sum_{k=0}^{r+s-i+1} \binom{r+s-i+1}{k} \Big[\sum_{j=1}^{\infty} \frac{(-1)^{j+1} n^{k-j}}{j} + n^k \ln n \Big] \\
&= \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1+x)^i(-1)^{r+i+k+1}}{k(r+s-i+1)}. \qquad (11)
\end{aligned}$$

Next, since $I_2 = O(n^{-n})$, it follows that

$$\lim_{n\to\infty} I_2 = 0. \qquad (12)$$

Equation (9) now follows from equations (10) to (12).

Corollary 2.6 *The neutrix convolution* $(1-x)^s \ln(1+x_-) \circledast x^r$ *exists and*

$$(1-x)^s \ln(1+x_-) \circledast x^r = \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1-x)^i(-1)^{r+i+k+1}}{k(r+s-i+1)}, \qquad (13)$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (13) follows from equation (9) on replacing x by $-x$.

Corollary 2.7 *The neutrix convolution* $(1+x)^s \ln(1+x_+) \circledast x_-^r$ *exists and*

$$\begin{aligned}(1+x)^s \ln(1+x_+) \circledast x_-^r &= \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1+x)^i(-1)^{i+j+1}}{k(r+s-i+1)} \\ &\quad - \sum_{i=0}^{r} \binom{r}{i} (-1)^i \Big\{ \frac{(1+x)^{r+s+1} \ln(1+x_+)}{r+s-i+1} \\ &\quad - \frac{[H(x)+x_+]^{r+s+1} - [H(x)+x_+]^i}{(r+s-i+1)^2} \Big\} \qquad (14)\end{aligned}$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Using equations (1) and (9), we have

$$\begin{aligned}(1+x)^s \ln(1+x_+) \circledast x^r &= (1+x)^s \ln(1+x_+) \circledast [x_+^r + (-1)^r x_-^r] \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \Big\{ \frac{(1+x)^{r+s+1} \ln(1+x_+)}{r+s-i+1} \\ &\quad - \frac{[H(x)+x_+]^{r+s+1} - [H(x)+x_+]^i}{(r+s-i+1)^2} \Big\} \\ &\quad + (-1)^r (1+x)^s \ln(1+x_+) \circledast x_-^r\end{aligned}$$

and equation (14) follows.

Corollary 2.8 *The neutrix convolution* $(1-x)^s \ln(1+x_-) \circledast x_+^r$ *exists and*

$$\begin{aligned}(1-x)^s \ln(1+x_-) \circledast x_+^r &= \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1-x)^i(-1)^{i+j}}{k(r+s-i+1)} \\ &\quad - \sum_{i=0}^{r} \binom{r}{i} (-1)^i \Big\{ \frac{(1-x)^{r+s+1} \ln(1+x_-)}{r+s-i+1} \\ &\quad - \frac{[H(-x)+x_-]^{r+s+1} - [H(-x)+x_-]^i}{(r+s-i+1)^2} \Big\} \qquad (15)\end{aligned}$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (15) follows from equation (14) on replacing x by $-x$.

Theorem 2.9 *The neutrix convolution* $x^s \ln(1+x_+) \circledast x^r$ *exists and*

$$x^s \ln(1+x_+) \circledast x^r = \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \binom{r+k-i+1}{k} \frac{(-1)^{s-j+r-i+k+1}(1+x)^i}{k(r+j-i+1)}, \qquad (16)$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Putting $[x^s \ln(1+x_+)]_n = x^s \ln(1+x_+)\tau_n(x)$ and $u = 1+t$, we have

$$\begin{aligned}
[x^s \ln(1+x_+)]_n \circledast x^r &= \int_0^n t^s \ln(1+t)(x-t)^r\,dt \\
&\quad + \int_n^{n+n^{-n}} t^s \ln(1+t)(x-t)^r \tau_n(t)\,dt \\
&= \int_1^{n+1} (u-1)^s \ln u (1+x-u)^r\,dt \\
&\quad + \int_n^{n+n^{-n}} t^s \ln(1+t)(x-t)^r \tau_n(t)\,dt \\
&= J_1 + J_2, \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j+r-i}(1+x)^i \int_1^{1+n} u^{r+j-i}\ln u\,du \\
&= \sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j+r-i}(1+x)^i \Big[\frac{(1+n)^{r+j-i+1}\ln(1+n)}{r+j-i+1} \\
&\quad - \frac{(1+n)^{r+j-i+1}-1}{(r+j-i+1)^2}\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} J_1 &= \operatorname*{N-lim}_{n\to\infty}\sum_{i=0}^{r}\sum_{j=0}^{s}\binom{r}{i}\binom{s}{j}\frac{(1+x)^i(-1)^{s-j+r-i}}{r+j-i+1} \\
&\quad \times \sum_{k=0}^{r+j-i+1}\binom{r+j-i+1}{k}\Big[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}n^{k-m}}{m} + n^k \ln n\Big] \\
&= \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{k=1}^{r+j-i+1}\binom{r}{i}\binom{s}{j}\binom{r+k-i+1}{k}\frac{(-1)^{s-j+r-i+k+1}(1+x)^i}{k(r+j-i+1)}. \qquad (18)
\end{aligned}$$

It follows as above that

$$\lim_{n\to\infty} J_2 = 0 \qquad (19)$$

and equation (16) now follows from equations (17) to (19).

Corollary 2.10 *The neutrix convolution $x^s \ln(1+x_-) \circledast x_-^r$ exists and*

$$x^s \ln(1+x_-) \circledast x^r = \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{k=1}^{r+j-i+1}\binom{r}{i}\binom{r+k-i+1}{k}\frac{(1-x)^i(-1)^{j-i+k+1}}{k(r+j-i+1)}, \qquad (20)$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (20) follows from equation (16) on replacing x by $-x$.

Corollary 2.11 *The neutrix convolution $x^s \ln(1+x_+) \circledast x_-^r$ exists and*

$$\begin{aligned} x^s \ln(1+x_+) \circledast x_-^r &= \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{k=1}^{r+j-i+1}\binom{r}{i}\binom{s}{j}\binom{r+k-i+1}{k} \\ &\quad\times\frac{(-1)^{s-j-i+k+1}(1+x)^i}{k(r+j-i+1)} \\ &\quad-\sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j-i}\Big[\frac{(1+x)^{r+j+1}\ln(1+x_+)}{r+j-i+1} \\ &\quad-\frac{[H(x)+x_+]^{r+j+1}-[H(x)+x_+]^i}{(r+j-i+1)^2}\Big] \end{aligned} \tag{21}$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Using equations (5) and (16), we have

$$\begin{aligned} x^s \ln(1+x_+) \circledast x^r &= x^s \ln(1+x_+) \circledast [x_+^r + (-1)^r x_-^r] \\ &= \sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j+r-i}\Big[\frac{(1+x)^{r+j+1}\ln(1+x_+)}{r+j-i+1} \\ &\quad-\frac{[H(x)+x_+]^{r+j+1}-[H(x)+x_+]^i}{(r+j-i+1)^2}\Big] \\ &\quad+(-1)^r x^s \ln(1+x_+) \circledast x_-^r \end{aligned}$$

and equation (21) follows.

Corollary 2.12 *The neutrix convolution $x^s \ln(1+x_-) \circledast x_+^r$ exists and*

$$\begin{aligned} x^s \ln(1+x_-) \circledast x_+^r &= \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{k=1}^{r+j-i+1}\binom{r}{i}\binom{s}{j}\binom{r+k-i+1}{k} \\ &\quad\times\frac{(-1)^{j-i+k+1}(1-x)^i}{k(r+j-i+1)} \\ &\quad-\sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{s}\binom{s}{j}(-1)^{j-i}\Big[\frac{(1-x)^{r+j+1}\ln(1+x_-)}{r+j-i+1} \\ &\quad-\frac{[H(-x)+x_-]^{r+j+1}-[H(-x)+x_-]^i}{(r+j-i+1)^2}\Big], \end{aligned} \tag{22}$$

for $r, s = 0, 1, 2, \ldots$.

Proof. Equation (22) follows from equation (21) on replacing x by $-x$.

For further results on the neutrix convolution, see [3] and [4].

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Received: July, 2013