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The classifications of low-dimensional Hom-Lie triple systems

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Abstract

In this paper, we determined the two dimensional and three dimensional endomorphism of Lie triple systems on complex field using undetermined coefficients method, and then classified the Hom-Lie triple systems when the twisted map α is not equal to the identity map.

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1 Introduction

A Hom-Lie algebra is a vector space endowed with a skew symmetric bracket satisfying a Jacobi identity twisted by a map. Before Hom-Lie algebras appeared, Hu studied q-Lie algebras, which are special Hom-Lie algebras[3]. Lie algebras are special cases of Hom-Lie algebras when the twisted map is the identity map. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the q-deformation of the Witt and the Virasoro algebras[2]. Since then, Hom-type algebras have been investigated by many authors. In particular, the notion of Hom-Lie triple systems was introduced by Yau[7].

We have known the classification of low-dimensional Lie triple systems. We can determine the low-dimensional endomorphism of Lie triple systems by using undetermined coefficients method. And then we can classify the two dimensional Hom-Lie triple systems and three dimensional Hom-Lie triple systems when the twisted map α is a multiplicative map.

2 Preliminary Notes

We start by recalling the definitions of Lie triple systems and Hom-Lie triple systems.

Definition 2.1 *[4] A vector space T together with a trilinear map $(x, y, z) \mapsto [x, y, z]$ is called a Lie triple system (LTS for short) if*

(1) $[x, x, z] = 0$,

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

(3) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all $x, y, z, u, v \in T$.

Definition 2.2 *[7] A Hom-Lie triple system (Hom-LTS for short) $(T, [\cdot, \cdot, \cdot], \alpha = (\alpha_1, \alpha_2))$ consists of an $\mathbf{F}$-vector space T, a trilinear map $[\cdot, \cdot, \cdot] : T \times T \times T \to T$, and linear maps $\alpha_i : T \to T$ for $i = 1, 2$, called twisted maps, such that for all $x, y, z, u, v \in T$,*

(1) $[x, x, z] = 0$,

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

(3)
$$\begin{aligned}[\alpha_1(u), \alpha_2(v), [x, y, z]] = {} & [[u, v, x], \alpha_1(y), \alpha_2(z)] + [\alpha_1(x), [u, v, y], \alpha_2(z)] \\ & + [\alpha_1(x), \alpha_2(y), [u, v, z]].\end{aligned}$$

A Hom-Lie triple system is said to be multiplicative if $\alpha_1 = \alpha_2 = \alpha$ and $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$, and denoted by $(T, [\cdot, \cdot, \cdot], \alpha)$.

A morphism $f : (T, [\cdot, \cdot, \cdot], \alpha = (\alpha_1, \alpha_2)) \to (T', [\cdot, \cdot, \cdot]', \alpha' = (\alpha'_1, \alpha'_2))$ of Hom-Lie triple systems is a linear map satisfying $f([x, y, z]) = [f(x), f(y), f(z)]'$ and $f \circ \alpha_i = \alpha'_i \circ f$ for $i = 1, 2$. An isomorphism is a bijective morphism.

Remark 2.3 *When the twisted maps α_i are both equal to the identity map, a Hom-Lie triple system is a Lie triple system. So Lie triple systems are special examples of Hom-Lie triple systems. More results about the Hom-Lie triple system are referred to [7].*

Definition 2.4 *[7] Let $(T, [\cdot, \cdot, \cdot], \alpha)$ be a Hom-Lie triple system, a subspace $D \subset T$ is called a Hom-subsystem if $\alpha(D) \subset D$ and $[D, D, D] \subset D$. A subspace $D \subset T$ is called a Hom-ideal if and $\alpha(D) \subset D$ and $[D, T, T] \subset D$.*

Throughout this paper $\mathbf{F}$ denotes an arbitrary field and Hom-Lie triple systems are multiplicative.

3 Main Results

Lemma 3.1 *[1] $(T,[\cdot,\cdot,\cdot])$ is a 2-dimensional Lie triple system on complex field and $\{e_1,e_2\}$ is its basis. Then we can find the possibility of the following types*

(1) T is an Abelian Lie triple system,

(2) $[e_1,e_2,e_1]=0$, $[e_1,e_2,e_2]=e_1$,

(3) $[e_1,e_2,e_1]=e_1$, $[e_1,e_2,e_2]=e_2$.

Theorem 3.2 *$(T,[\cdot,\cdot,\cdot]_\alpha,\alpha)$ is a 2-dimensional Hom-Lie triple system on complex field and $\{e_1,e_2\}$ is its basis. Then we can find the possibility of the following types, when the twisted map α is not equal to the identity map,*

(1) $(T,[\cdot,\cdot,\cdot]_\alpha,\alpha)$ is an Abelian Hom-Lie triple system,

(2) $[\alpha(e_1),\alpha(e_2),\alpha(e_1)]=0$, $[\alpha(e_1),\alpha(e_2),\alpha(e_2)]=\lambda_1e_1$, $\lambda_1\neq 0$,

(3) $[\alpha(e_1),\alpha(e_2),\alpha(e_1)]=\lambda_1e_1$, $[\alpha(e_1),\alpha(e_2),\alpha(e_2)]=-\frac{1}{\lambda_1}e_2$, $\lambda_1\neq 0$,

(4) $[\alpha(e_1),\alpha(e_2),\alpha(e_1)]=\lambda_2e_2$, $[\alpha(e_1),\alpha(e_2),\alpha(e_2)]=-\frac{1}{\lambda_2}e_1$, $\lambda_2\neq 0$.

Proof. We suppose that $\alpha(e_1)=\lambda_1e_1+\lambda_2e_2$,$\alpha(e_2)=\beta_1e_1+\beta_2e_2$, $A=\begin{pmatrix}\lambda_1 & \beta_1\\ \lambda_2 & \beta_2\end{pmatrix}$.

(1) $[\alpha(e_1),\alpha(e_2),\alpha(e_1)]=0$, $[\alpha(e_1),\alpha(e_2),\alpha(e_2)]=0$. Thus, $(T,[\cdot,\cdot,\cdot]_\alpha,\alpha)$ is an Abelian Hom-Lie triple system.

(2) We have

$$\begin{aligned}
& [\alpha(e_1),\alpha(e_2),\alpha(e_1)]=0\\
= & [\lambda_1e_1+\lambda_2e_2,\beta_1e_1+\beta_2e_2,\lambda_1e_1+\lambda_2e_2]\\
= & \lambda_1\beta_2\lambda_1[e_1,e_2,e_1]+\lambda_1\beta_2\lambda_2[e_1,e_2,e_2]+\lambda_2\beta_1\lambda_1[e_2,e_1,e_1]+\lambda_2\beta_1\lambda_2[e_2,e_1,e_2]\\
= & (\lambda_1\beta_2\lambda_2-\lambda_2\beta_1\lambda_2)[e_1,e_2,e_2]\\
= & (\lambda_1\beta_2\lambda_2-\lambda_2\beta_1\lambda_2)e_1.
\end{aligned}$$

and

$$\begin{aligned}
& [\alpha(e_1),\alpha(e_2),\alpha(e_2)]=\alpha(e_1)=\lambda_1e_1+\lambda_2e_2\\
= & (\lambda_1\beta_2\beta_2-\lambda_2\beta_1\beta_2)[e_1,e_2,e_2]\\
= & (\lambda_1\beta_2\beta_2-\lambda_2\beta_1\beta_2)e_1.
\end{aligned}$$

So, we can obtain

$$\begin{cases}\lambda_2(\lambda_1\beta_2-\lambda_2\beta_1)=0\\ \beta_2(\lambda_1\beta_2-\lambda_2\beta_1)=\lambda_1\\ \lambda_2=0.\end{cases}$$

That is

$$\begin{cases} \lambda_2 = 0 \\ \lambda_1\beta_2\beta_2 = \lambda_1. \end{cases}$$

We can get two types

a. $\lambda_1 = 0$, then β_1, β_2 can take all elements in T, $A = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}$.

b. $\lambda_1 \neq 0$, then $\beta_2 = \pm 1$, β_1 can take all elements in T, $A = \begin{pmatrix} \lambda_1 & \beta_1 \\ 0 & \pm 1 \end{pmatrix}$.

Thus, we obtain a type of the classification of 2-dimensional Hom-Lie triple systems $[\alpha(e_1), \alpha(e_2), \alpha(e_1)] = 0$, $[\alpha(e_1), \alpha(e_2), \alpha(e_2)] = \lambda_1 e_1$, $\lambda_1 \neq 0$.

(3) Using the same method which is used in (2). Thus, we obtain two types of the classification of 2-dimensional Hom-Lie triple systems

(i) $[\alpha(e_1), \alpha(e_2), \alpha(e_1)] = \lambda_1 e_1$, $[\alpha(e_1), \alpha(e_2), \alpha(e_2)] = -\frac{1}{\lambda_1} e_2$, $\lambda_1 \neq 0$.

(ii) $[\alpha(e_1), \alpha(e_2), \alpha(e_1)] = \lambda_2 e_2$, $[\alpha(e_1), \alpha(e_2), \alpha(e_2)] = -\frac{1}{\lambda_2} e_1$, $\lambda_2 \neq 0$.

□

Lemma 3.3 *[1] $(T, [\cdot,\cdot,\cdot])$ is a 3-dimensional Lie triple system on complex field and $\{e_1, e_2, e_3\}$ is its basis. Then we can find the possibility of the following types*

(1) T is an Abelian Lie triple system,

(2) T is a simple Lie triple system,

(3) $[e_2, e_3, e_3] = e_2$,

(4) $[e_1, e_2, e_1] = e_3$,

(5) $[e_1, e_3, e_3] = e_1$, $[e_2, e_3, e_3] = e_1$,

(6) $[e_1, e_2, e_1] = e_1$, $[e_1, e_2, e_2] = -e_2$,

the others are zero.

Theorem 3.4 *$(T, [\cdot,\cdot,\cdot]_\alpha, \alpha)$ is a 3-dimensional Hom-Lie triple system on complex field and $\{e_1, e_2, e_3\}$ is its basis. Then we can find the possibility of the following types, when the twisted map α is not equal to the identity map,*

(1) $(T, [\cdot,\cdot,\cdot]_\alpha, \alpha)$ is an Abelian Hom-Lie triple system,

(2) $(T, [\cdot,\cdot,\cdot]_\alpha, \alpha)$ is a simple Hom-Lie triple system,

(3) $[\alpha(e_2), \alpha(e_3), \alpha(e_3)] = \beta_2 e_2$, $\beta_2 \neq 0$,

(4) $[\alpha(e_1), \alpha(e_2), \alpha(e_3)] = \lambda_1^2 \beta_2 e_3$, $\lambda_1\beta_2 \neq 0$,

(5) $[\alpha(e_1), \alpha(e_3), \alpha(e_3)] = (\beta_1 + \beta_2)e_1$, $[\alpha(e_2), \alpha(e_3), \alpha(e_3)] = (\beta_1 + \beta_2)e_1$, $\beta_1 + \beta_2 \neq 0$,

(6) $[\alpha(e_1), \alpha(e_2), \alpha(e_1)] = \lambda_1 e_1$, $[\alpha(e_1), \alpha(e_2), \alpha(e_2)] = -\frac{1}{\lambda_1} e_2$, $\lambda_1 \neq 0$,

(7) $[\alpha(e_1), \alpha(e_2), \alpha(e_1)] = \lambda_2 e_2$, $[\alpha(e_1), \alpha(e_2), \alpha(e_2)] = -\frac{1}{\lambda_2} e_1$, $\lambda_2 \neq 0$,

the others are zero.

Proof. We can obtain the results in the same way which is used in Theorem 2.2. □

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References

[1] Bouetou, T., Classification of solvable 3-dimensional Lie triple systems, Non-associative algebra and its applications, (2003), 41-54.

[2] Hartwig, J.; Larsson, D.; Silvestrov, S., Deformations of Lie algebras using σ-derivations, Deformations of Lie algebras using σ-derivations, 295 (2006), no. 2, 314-361.

[3] Hu, N., Q-Witt algebras, q-Lie algebras, q-holomorph structure and representations, Algebra Coll, 6 (1999), 51-70.

[4] Lister, W., A structure theory of Lie triple systems, Trans. Amer. Math. Soc., 72 (1952), 217-242.

[5] Makhlouf, A.; Silvestrov, S., Hom-algebra structures, arXiv:math/ 0609501v3.

[6] Xu, L.; Wang, C.; Zhang, R.; Zhang, Q., Classification of Low-Dimensional Hom-Leibniz Algebras, Journal of Jilin University (Science Edition), (1) 51 (2013), 74-82.

[7] Yau, D., On n-ary Hom-Nambu and Hom-Nambu-Lie algebras, J. Geom. Phys., 62 (2012), 506-522.

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