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The asymptotics of a generalised Struve function

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Abstract

A generalised Struve function has recently been introduced by Ali, Mondal and Nisar [J. Korean Math. Soc. **54** (2017) 575–598] as

$$(\tfrac{1}{2}z)^{\nu+1}\sum_{n=0}^\infty \frac{(\frac{1}{2}z)^{2n}}{\Gamma(n+\frac{3}{2})\Gamma(an+\nu+\frac{3}{2})},$$

where a is a positive integer. In this paper, we obtain the asymptotic expansions of this function for large complex z when a is a real parameter satisfying $a>-1$. Some numerical examples are presented to confirm the accuracy of the expansions.

Mathematics Subject Classification: 30E15, 33E20, 34E05, 41A60

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1. Introduction The modified Struve function $\mathbf{L}_\nu(z)$ is a particular solution of the inhomogeneous modified Bessel equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\Big(1+\frac{\nu^2}{z^2}\Big)y=\frac{(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}$$

which possesses the series expansion [4, p. 288]

$$\mathbf{L}_\nu(z)=(\tfrac{1}{2}z)^{\nu+1}\sum_{n=0}^\infty \frac{(\frac{1}{2}z)^{2n}}{\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})} \tag{1.1}$$

valid for all finite z. The ordinary Struve function $\mathbf{H}_\nu(z)$ is given by the alternating version of (1.1) corresponding to z situated on the imaginary axis, and

$$\mathbf{H}_\nu(z)=\mp ie^{\mp\pi i\nu/2}\mathbf{L}_\nu(\pm iz). \tag{1.2}$$

A generalisation of the series (1.1) and its alternating version has been considered by Yagmur and Orban [13] by replacing the $\frac{3}{2}$ in the second gamma function by an arbitrary complex parameter. These authors determined sufficient conditions for it to be univalent and obtained various convexity properties of the functions. In a recent paper by Ali, Mondal and Nisar [1], the above series was generalised further by the introduction of a multiple argument in the second gamma function, namely the function

$$\sum_{n=0}^\infty \frac{(\pm 1)^n (\frac{1}{2}z)^{2n+\nu+1}}{\Gamma(n+\frac{3}{2})\Gamma(an+\nu+\frac{3}{2})},$$

where a denotes a positive integer. These authors showed that this function is a solution of an $(a+1)$th-order differential equation. They also investigated its monotonicity and log-convexity properties and established Turán-type inequalities. Upper bounds satisfied by this function in the case $a=2$ and for the non-alternating series were derived.

In this paper we consider the generalised Struve function defined in [1] by

$$\mathbf{L}_\nu(z;a) = (\tfrac{1}{2}z)^{\nu+1} \sum_{n=0}^\infty \frac{(\frac{1}{2}z)^{2n}}{\Gamma(n+\frac{3}{2})\Gamma(an+\nu+\frac{3}{2})} \tag{1.3}$$

where a is a real parameter satisfying $a > -1$ for convergence of the series and we restrict ν to be real. It is then evident that

$$\mathbf{L}_\nu(ze^{\pm\pi i};a) = e^{\pm\pi i(\nu+1)}\mathbf{L}_\nu(z;a), \qquad \mathbf{L}_\nu(\overline{z};a) = \overline{\mathbf{L}_\nu(z;a)},$$

where the bar denotes the complex conjugate, so that we may confine our attention to the sector $0 \le \arg z \le \frac{1}{2}\pi$. When $\arg z = \frac{1}{2}\pi$ in (1.3), we generate the generalised alternating version given by $\mathbf{H}_\nu(|z|;a)$. We determine the asymptotic expansion of $\mathbf{L}_\nu(z;a)$ for large complex z and finite values of ν and a. The series in (1.3) is a particular case of a generalised Wright function; see (2.1) below. Accordingly, we employ the well-established asymptotic theory of the Wright function to determine the large-z expansion of $\mathbf{L}_\nu(z;a)$, a summary of which is presented in Section 2. It will be found that the analysis of $\mathbf{L}_\nu(z;a)$ separates into two distinct cases according as $a > 0$ and $-1 < a < 0$.

2. Standard theory of the generalised Wright function The generalised Wright function is defined by the series

$${}_p\Psi_q(z) \equiv {}_p\Psi_q\left(\begin{matrix}(\alpha_1,a_1),\ldots,(\alpha_p,a_p)\\(\beta_1,b_1),\ldots,(\beta_q,b_q)\end{matrix}\middle| z\right) = \sum_{n=0}^\infty g(n)\,\frac{z^n}{n!}, \tag{2.1}$$

$$g(n) = \frac{\prod_{r=1}^p \Gamma(\alpha_r n + a_r)}{\prod_{r=1}^q \Gamma(\beta_r n + b_r)}, \tag{2.2}$$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. We also assume that the α_r and a_r are subject to the restriction

$$\alpha_r n + a_r \neq 0, -1, -2, \ldots \qquad (n = 0, 1, 2, \ldots\ ;\ 1 \leq r \leq p) \tag{2.3}$$

so that no gamma function in the numerator in (2.1) is singular.

We introduce the parameters associated[1] with $g(n)$ given by

$$\kappa = 1 + \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r, \qquad h = \prod_{r=1}^{p} \alpha_r^{\alpha_r} \prod_{r=1}^{q} \beta_r^{-\beta_r},$$

$$\vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}(q-p), \qquad \vartheta' = 1 - \vartheta. \tag{2.4}$$

If it is supposed that α_r and β_r are such that $\kappa > 0$ then ${}_p\Psi_q(z)$ is uniformly and absolutely convergent for all finite z. If $\kappa = 0$, the sum in (1.1) has a finite radius of convergence equal to h^{-1}, whereas for $\kappa < 0$ the sum is divergent for all nonzero values of z. The parameter κ will be found to play a critical role in the asymptotic theory of ${}_p\Psi_q(z)$ by determining the sectors in the z-plane in which its behaviour is either exponentially large, algebraic or exponentially small in character as $|z| \to \infty$.

The determination of the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$ and finite values of the parameters has a long history. Detailed investigations were carried out by Wright [10, 11] and by Braaksma [3] for a more general class of integral functions than (1.1). We present below a summary of the main expansion theorems related to the asymptotics of ${}_p\Psi_q(z)$ for large $|z|$; for a recent presentation, see [7]. In order to do this we first introduce the exponential expansion $E_{p,q}(z)$ and the algebraic expansion $H_{p,q}(z)$ associated with ${}_p\Psi_q(z)$.

The exponential expansion $E_{p,q}(z)$ is given by the formal asymptotic sum

$$E_{p,q}(z) := Z^{\vartheta} e^{Z} \sum_{j=0}^{\infty} A_j Z^{-j}, \qquad Z = \kappa (hz)^{1/\kappa}, \tag{2.5}$$

where the coefficients A_j are those appearing in the inverse factorial expansion of $g(s)/s!$ given by

$$\frac{g(s)}{\Gamma(1+s)} = \kappa (h\kappa^{\kappa})^{s} \left\{ \sum_{j=0}^{M-1} \frac{A_j}{\Gamma(\kappa s + \vartheta' + j)} + \frac{\rho_M(s)}{\Gamma(\kappa s + \vartheta' + M)} \right\}. \tag{2.6}$$

Here $g(s)$ is defined in (2.2) with n replaced by s, M is a positive integer and $\rho_M(s) = O(1)$ for $|s| \to \infty$ in $|\arg s| < \pi$. The leading coefficient A_0 is

[1] Empty sums and products are to be interpreted as zero and unity, respectively.

specified by

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\kappa^{-\frac{1}{2}-\vartheta}\prod_{r=1}^{p}\alpha_r^{a_r-\frac{1}{2}}\prod_{r=1}^{q}\beta_r^{\frac{1}{2}-b_r}. \tag{2.7}$$

The coefficients A_j are independent of s and depend only on the parameters p, q, α_r, β_r, a_r and b_r. An algorithm for their evaluation is described in [7, 8].

The algebraic expansion $H_{p,q}(z)$ follows from the Mellin-Barnes integral representation [9, §2.4]

$$ {}_p\Psi_q(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(s)g(-s)(ze^{\mp\pi i})^{-s}ds, \qquad |\arg(-z)| < \pi(1-\tfrac{1}{2}\kappa), \tag{2.8}$$

where the path of integration is indented near $s = 0$ to separate[2] the poles of $\Gamma(s)$ at $s = -k$ from those of $g(-s)$ situated at

$$s = (a_r + k)/\alpha_r, \qquad k = 0, 1, 2, \ldots \ (1 \le r \le p). \tag{2.9}$$

In general there will be p such sequences of simple poles though, depending on the values of α_r and a_r, some of these poles could be multiple poles or even ordinary points if any of the $\Gamma(\beta_r s + b_r)$ are singular there. Displacement of the contour to the right over the poles of $g(-s)$ then yields the algebraic expansion of ${}_p\Psi_q(z)$ valid in the sector in (2.8).

If it is assumed that the parameters are such that the poles in (2.9) are all simple we obtain the algebraic expansion given by $H_{p,q}(z)$, where

$$H_{p,q}(z) := \sum_{m=1}^{p}\alpha_m^{-1}z^{-a_m/\alpha_m}S_{p,q}(z;m) \tag{2.10}$$

and $S_{p,q}(z;m)$ denotes the formal asymptotic sum

$$S_{p,q}(z;m) := \sum_{k=0}^{\infty}\frac{(-)^k}{k!}\Gamma\left(\frac{k+a_m}{\alpha_m}\right)\frac{\prod_{r=1}^{\prime p}\Gamma(a_r-\alpha_r(k+a_m)/\alpha_m)}{\prod_{r=1}^{q}\Gamma(b_r-\beta_r(k+a_m)/\alpha_m)}z^{-k/\alpha_m}, \tag{2.11}$$

with the prime indicating the omission of the term corresponding to $r = m$ in the product. This expression in (2.10) consists of (at most) p expansions each with the leading behaviour z^{-a_m/α_m} $(1 \le m \le p)$. When the parameters α_r and a_r are such that some of the poles are of higher order, the expansion (2.11) is invalid and the residues must then be evaluated according to the multiplicity of the poles concerned; this will lead to terms involving $\log z$ in the algebraic expansion.

The expansion theorems for ${}_p\Psi_q(z)$ are as follows. Throughout we let ϵ denote an arbitrarily small positive quantity.

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[2]This is always possible when the condition (2.3) is satisfied.

Theorem 1. *When $0 < \kappa \leq 2$, then*

$$
{}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & in \quad |\arg z| \leq \min\{\pi - \epsilon, \pi\kappa - \epsilon\} \\ \\ H_{p,q}(ze^{\mp\pi i}) & in \quad \pi\kappa + \epsilon \leq |\arg z| \leq \pi \quad (0 < \kappa < 1) \\ \\ E_{p,q}(z) + E_{p,q}(ze^{\mp 2\pi i}) & \\ +H_{p,q}(ze^{\mp\pi i}) & in \quad |\arg z| \leq \pi \quad (1 < \kappa \leq 2) \end{cases} \tag{2.12}
$$

as $|z| \to \infty$. The upper or lower signs are chosen according as $\arg z > 0$ *or* $\arg z < 0$*, respectively.*

Theorem 2. *When $\kappa > 2$ we have*

$$
{}_p\Psi_q(z) \sim \sum_{n=-N}^{N} E_{p,q}(ze^{2\pi i n}) + H_{p,q}(ze^{\mp\pi i z}) \tag{2.13}
$$

as $|z| \to \infty$ in the sector $|\arg z| \leq \pi$. The integer N is chosen such that it is the smallest integer satisfying $2N + 1 > \frac{1}{2}\kappa$ and the upper or lower is chosen according as $\arg z > 0$ *or* $\arg z < 0$*, respectively.*

In this case the asymptotic behaviour of ${}_p\Psi_q(z)$ is exponentially large for all values of $\arg z$ and, consequently, the algebraic expansion may be neglected. The sums $E_{p,q}(ze^{2\pi i n})$ are exponentially large (or oscillatory) as $|z| \to \infty$ for values of $\arg z$ satisfying $|\arg z + 2\pi n| \leq \frac{1}{2}\pi\kappa$.

The division of the z-plane into regions where ${}_p\Psi_q(z)$ possesses exponentially large or algebraic behaviour for large $|z|$ is illustrated in Fig. 1. When $0 < \kappa < 2$, the exponential expansion $E_{p,q}(z)$ is still present in the sectors $\frac{1}{2}\pi\kappa < |\arg z| < \min\{\pi, \pi\kappa\}$, where it is subdominant. The rays $\arg z = \pm\pi\kappa$ $(0 < \kappa < 1)$, where $E_{p,q}(z)$ is *maximally* subdominant with respect to $H_{p,q}(ze^{\mp\pi i})$, are called Stokes lines.[3] As these rays are crossed (in the sense of increasing $|\arg z|$) the exponential expansion switches off according to Berry's now familiar error-function smoothing law [2]; see [5] for details. The rays $\arg z = \pm\frac{1}{2}\pi\kappa$, where $E_{p,q}(z)$ is oscillatory and comparable to $H_{p,q}(ze^{\mp\pi i})$, are called anti-Stokes lines.

We omit the expansion *on* the Stokes lines $\arg z = \pm\pi\kappa$ in Theorem 1; the details in the case $p = 1$, $q \geq 0$ are discussed in [6]; see also [8] for the case of the generalised Bessel function. Since $E_{p,q}(z)$ is exponentially small in $\frac{1}{2}\pi\kappa < |\arg z| \leq \pi$, then in the sense of Poincaré, the expansion $E_{p,q}(z)$ can be neglected in these sectors. Similarly, $E_{p,q}(ze^{-2\pi i})$ is exponentially small compared to $E_{p,q}(z)$ in $0 \leq \arg z < \pi$ and therefore can be neglected when

[3]The positive real axis $\arg z = 0$ is also a Stokes line where the algebraic expansion is maximally subdominant.

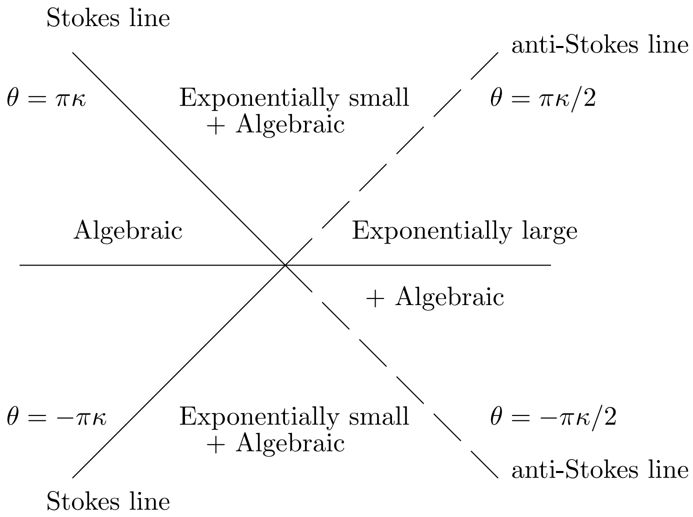

Figure 1: The exponentially large and algebraic sectors associated with ${}_p\Psi_q(z)$ in the complex z-plane with $\theta = \arg z$ when $0 < \kappa < 1$. The Stokes and anti-Stokes lines are indicated.

$1 < \kappa < 2$. However, in the vicinity of $\arg z = \pi$, these last two expansions are of comparable magnitude and, for real parameters, they combine to generate a real result on this ray. A similar remark applies to $E_{p,q}(ze^{2\pi i})$ in $-\pi < \arg z \leq 0$.

3. The asymptotic expansion of $\mathbf{L}_\nu(z;a)$ when $a > 0$ The function $\mathbf{L}_\nu(z;a)$ can be written as

$$(\tfrac{1}{2}z)^{-\nu-1}\,\mathbf{L}_\nu(z;a) = {}_1\Psi_2\left(\begin{matrix}(1,1)\\(1,\frac{3}{2}),(a,\nu+\frac{3}{2})\end{matrix}\,\middle|\,\zeta\right) \equiv {}_1\Psi_2(\zeta), \qquad \zeta = \tfrac{1}{4}z^2, \quad (3.1)$$

where the Wright function ${}_1\Psi_2(\zeta)$ is defined in (2.1). From (2.4) and (2.7) we have the parameters associated with the right-hand side of (3.1)

$$\kappa = 1 + a, \qquad h = a^{-a}, \qquad \vartheta = -\nu - \tfrac{3}{2}, \qquad A_0 = \frac{1}{\sqrt{2\pi}}\,(\kappa/a)^{\nu+1}.$$

We note that $\kappa > 1$ when $a > 0$.

The algebraic and exponential expansions associated with ${}_1\Psi_2(\zeta)$ are, from (2.5) and (2.10),

$$H_{1,2}(\zeta) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{1}{2})\,\zeta^{-k-1}}{\Gamma(\nu+\frac{3}{2}-a(1+k))}$$

and

$$E_{1,2}(\zeta) = A_0 Z^{\vartheta} e^{Z}\sum_{j=0}^{\infty} c_j Z^{-j}, \qquad Z = \kappa(h\zeta)^{1/\kappa},$$

where $c_0 = 1$. The coefficients $c_j \equiv c_j(a,\nu)$ $(j \geq 1)$ can be determined by the algorithm described in [7, Appendix]; see also [8], [9, p. 46]. It is found that

$$c_1(a,\nu) = -\frac{1}{24a}\{11 + 24\nu + 12\nu^2 - a(25 + 24\nu) + 11a^2\},$$

$$c_2(a,\nu) = \frac{1}{1152a^2}\{265 + 1056\nu + 1416\nu^2 + 768\nu^3 + 144\nu^4 - 2a(791 + 2040\nu$$
$$+1596\nu^2 + 384\nu^3) + 3a^2(905 + 1360\nu + 472\nu^2) - 2a^3(791 + 528\nu) + 265a^4\},$$

$$c_3(a,\nu) = -\frac{1}{414720a^3}\{(48703+286200\nu+617940\nu^2+636480\nu^3+334800\nu^4+86400\nu^5$$
$$+8640\nu^6)-3a(189797+791400\nu+1179240\nu^2+797760\nu^3+248400\nu^4+28800\nu^5)$$
$$+6a^2(355459 + 1019700\nu + 996570\nu^2 + 398880\nu^3 + 55800\nu^4)$$
$$-a^3(3254507 + 6118200\nu + 3537720\nu^2 + 636480\nu^3)$$
$$+6a^4(355459+395700\nu+102990\nu^2)-3a^5(189797+95400\nu)+48703a^6\}. \quad (3.2)$$

The complexity of the higher coefficients prevents their presentation in general form. However, in specific cases, where the parameters have numerical values, it is possible to generate many coefficients; see the example in Section 5.

Then from Theorems 1 and 2 we obtain

Theorem 3. *When $a > 0$ and $\zeta = z^2/4$, we have the expansion of the generalised Struve function*

$$(\tfrac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;a) \sim \begin{cases} E_{1,2}(\zeta) + E_{1,2}(\zeta e^{\mp 2\pi i}) + H_{1,2}(\zeta e^{\mp\pi i}) & (1 < \kappa \leq 2) \\ \displaystyle\sum_{n=-N}^{N} E_{1,2}(\zeta e^{2\pi i n}) + H_{1,2}(\zeta e^{\mp\pi i}) & (\kappa > 2) \end{cases} \quad (3.3)$$

as $|z| \to \infty$ in the sector $|\arg z| \leq \frac{1}{2}\pi$. The integer N is chosen such that it is the smallest integer satisfying $2N + 1 > \frac{1}{2}\kappa$ and the upper or lower signs are chosen according as $\arg z > 0$ *or* $\arg z < 0$, *respectively.*

The expansion $E_{1,2}(\zeta e^{-2\pi i})$ is exponentially smaller than $E_{1,2}(\zeta)$ in $0 \leq \arg\zeta < \pi$ $(0 \leq \arg z < \frac{1}{2}\pi)$ and can be neglected, depending on the accuracy required, but becomes comparable to $E_{1,2}(\zeta)$ in the vicinity of $\arg\zeta = \pi$. A similar remark applies to $E_{1,2}(\zeta e^{2\pi i})$ in the lower half-plane. In fact, the expansions $E_{1,2}(\zeta e^{\mp 2\pi i})$ switch off as $|\arg\zeta|$ decreases across the Stokes lines for these functions; see [7, §3.4] for a numerical example. Since the exponential factors associated with $E_{1,2}(\zeta)$ and $E_{1,2}(\zeta e^{\mp 2\pi i})$ are $\exp[|Z|e^{i\phi/\kappa}]$ and $\exp[|Z|e^{i(\phi\mp 2\pi)/\kappa}]$, where $\phi = \arg\zeta$, the greatest difference between the real parts of these factors occurs when $\sin(\phi/\kappa) = \sin((\phi \mp 2\pi)/\kappa)$; that is, on the

Stokes lines $\phi = \pm\pi(1-\frac{1}{2}\kappa)$. Consequently, when $1<\kappa<2$, the expansions $E_{1,2}(\zeta e^{\mp 2\pi i})$ are not present as $|z|\to\infty$ in the sector $|\arg z| < \frac{1}{2}\pi(1-\frac{1}{2}\kappa)$.

When $z>0$ ($\arg\zeta = 0$), the algebraic expansion in (3.3) is maximally subdominant as $z\to+\infty$ and undergoes a Stokes phenomenon. Neglecting this subdominant contribution, we therefore have

$$(\tfrac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;a) \sim E_{1,2}(\zeta) \qquad (1<\kappa\le 2;\ z\to+\infty). \tag{3.4}$$

When $\arg z = \frac{1}{2}\pi$ ($\arg\zeta = \pi$) we have from (3.3) with $\zeta = xe^{\pi i}$ the expansion

$$\begin{aligned}(\tfrac{1}{2}i|z|)^{-\nu-1}\mathbf{L}_\nu(i|z|;a) &\sim E_{1,2}(xe^{\pi i}) + E_{1,2}(xe^{-\pi i}) + H_{1,2}(x)\\ &= \tilde{E}_1(X) + H_{1,2}(x) \qquad (1<\kappa\le 2)\end{aligned} \tag{3.5}$$

as $|z|\to+\infty$, where

$$\tilde{E}_n(X) := 2A_0X^\vartheta e^{X\cos(2n-1)\pi/\kappa}\sum_{j=0}^\infty c_jX^{-j}\cos\left[X\sin\frac{(2n-1)\pi}{\kappa}+\frac{\pi}{\kappa}(\vartheta-j)\right]$$

for positive integer n and

$$x = \tfrac{1}{4}|z|^2, \qquad X = \kappa(hx)^{1/\kappa}.$$

The algebraic expansion $H_{1,2}(x)$ is dominant as $|z|\to\infty$.

When $\kappa>2$, the expansion of $(\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;a)$ as $z\to+\infty$ is given by the second expression in (3.3), where the maximally subdominant algebraic expansion may be neglected. When $z = i|z|$, we have

$$(\tfrac{1}{2}i|z|)^{-\nu-1}\mathbf{L}_\nu(i|z|;a) \sim \sum_{n=1}^N \tilde{E}_n(X) + H_{1,2}(x) \tag{3.6}$$

as $|z|\to\infty$, where N is as specified above and we have omitted the exponentially small contribution represented by $E_{1,2}(xe^{(2N+1)\pi i})$. The growth of $\mathbf{L}_\nu(i|z|;a)$ in this case is exponentially large as $|z|\to\infty$.

4. The asymptotic expansion of $\mathbf{L}_\nu(z;a)$ when $-1<a<0$ When $-1<a<0$ we write $a=-\sigma$, where $0<\sigma<1$. Then use of the reflection formula for the gamma function shows that

$$\begin{aligned}(\tfrac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\sigma) &= \frac{1}{\pi}\sum_{n=0}^\infty \frac{\Gamma(\sigma n-\nu-\frac{1}{2})\,(\frac{1}{2}z)^{2n}}{\Gamma(n+\frac{3}{2})}\sin\pi(-\sigma n+\nu+\tfrac{3}{2})\\ &= \frac{i}{2\pi}\Big\{e^{\pi i\vartheta}{}_2\Psi_1(\zeta e^{\pi i\sigma}) - e^{-\pi i\vartheta}{}_2\Psi_1(\zeta e^{-\pi i\sigma})\Big\},\end{aligned} \tag{4.1}$$

where $\zeta = z^2/4$ and $\vartheta = -\nu - \frac{3}{2}$ as in Section 3 and

$$ {}_2\Psi_1(\zeta) \equiv {}_2\Psi_1\left(\begin{array}{c} (1,1), (\sigma, -\nu - \frac{1}{2}) \\ (1, \frac{3}{2}) \end{array} \middle| \zeta \right) = \sum_{n=0}^{\infty} \frac{\Gamma(\sigma n - \nu - \frac{1}{2})\, \zeta^n}{\Gamma(n + \frac{3}{2})} \tag{4.2} $$

provided $\Gamma(\sigma n - \nu - \frac{1}{2})$ is regular for $n = 0, 1, 2, \ldots$.

The parameters associated with the Wright function in (4.2) are

$$ \kappa = 1 - \sigma, \qquad h = \sigma^{\sigma}, \qquad A_0 = \sqrt{2\pi}\, (\kappa/\sigma)^{\nu+1}. \tag{4.3} $$

Then, since $0 < \kappa < 1$, we have from Theorem 1

$$ {}_2\Psi_1(\zeta) \sim \begin{cases} E_{2,1}(\zeta) + H_{2,1}(\zeta e^{\mp\pi i}) & \text{in } |\arg \zeta| \leq \pi\kappa - \epsilon \\ \\ H_{2,1}(\zeta e^{\mp\pi i}) & \text{in } \pi\kappa + \epsilon \leq |\arg \zeta| \leq \pi \end{cases} \tag{4.4} $$

as $\zeta \to \infty$, where from (2.5) the exponential expansion is

$$ E_{2,1}(\zeta) = A_0 Z^{\vartheta} e^{Z} \sum_{j=0}^{\infty} c_j Z^{-j}, \qquad Z = \kappa (h\zeta)^{1/\kappa}, \tag{4.5} $$

with the coefficients $c_j = c_j(-\sigma, \nu)$ given in (3.2); see the appendix. From (2.10), the algebraic expansion consists of two asymptotic series, viz.

$$ H_{2,1}(\zeta) = \sum_{k=0}^{\infty} (-)^k G_k \zeta^{-k-1} + \frac{1}{\sigma} \sum_{k=0}^{\infty} (-)^k G'_k \zeta^{-k_s}, \quad k_s := \frac{1}{\sigma}(k - \nu - \tfrac{1}{2}), \tag{4.6} $$

where, provided k_s is not equal to an integer[4],

$$ G_k = \frac{\Gamma(-\nu - \frac{1}{2} - \sigma(k+1))}{\Gamma(\frac{1}{2} - k)}, \qquad G'_k = \frac{\Gamma(k_s)\Gamma(1 - k_s)}{k!\, \Gamma(\frac{3}{2} - k_s)}. $$

The expansion of ${}_2\Psi_1(\zeta)$ on the Stokes lines $\arg \zeta = \pm\pi\kappa$, where the exponential expansion $E_{2,1}(\zeta)$ is maximally subdominant and is in the process of switching off (as $|\arg \zeta|$ increases), is omitted here. This has been considered for the more general function ${}_1\Psi_q(z)$ for integer $q \geq 0$ in [6, §5] and for the generalised Bessel function ${}_0\Psi_1(z)$ in [8].

The expansion of $\mathbf{L}_{\nu}(z; -\sigma)$ can be constructed from knowledge of the expansion of the associated function ${}_2\Psi_1(\zeta)$ in (4.4). To keep the presentation as clear as possible, we restrict our attention here to the most commonly occurring situation of real ζ (that is, $z > 0$ or $z = i|z|$) in (4.1).

4.1. The expansion for $z \to +\infty$

[4]When $k = 1, 2, \ldots$ there are double poles in the integrand of (2.8); the values $k_s = 0, -1, -2, \ldots$ are disallowed by (2.9).

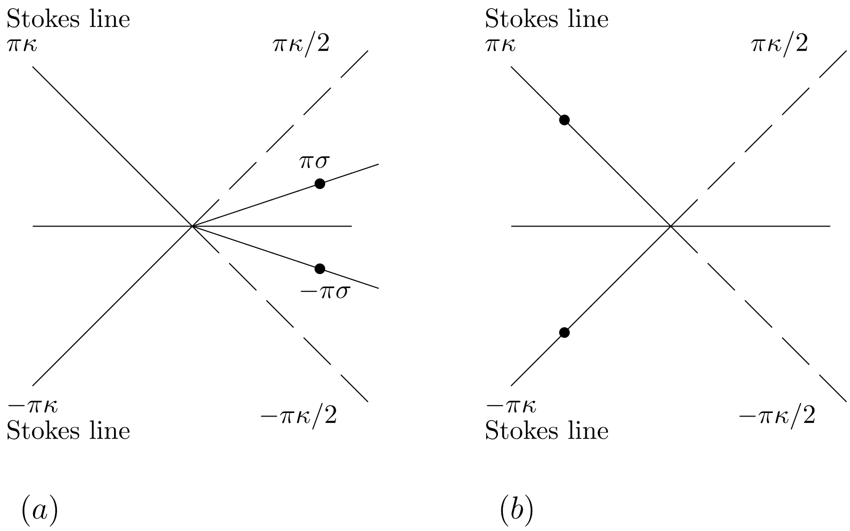

Figure 2: The Stokes and anti-Stokes lines when $0<\kappa<1$ and the location of the arguments $P_\pm$ (indicated by heavy dots): (a) $P_\pm = xe^{\pm\pi i\sigma}$ and (b) $P_\pm = xe^{\pm\pi i\kappa}$.

When $z>0$ ($\zeta>0$) we write $\zeta=x$. From (4.1) and (4.6), the algebraic component of the expansion of $(\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\sigma)$ is

$$\hat{H}_{2,1}(x) \equiv \frac{i}{2\pi}\Big\{e^{\pi i\vartheta}H_{2,1}(xe^{\pi i\sigma}\cdot e^{-\pi i}) - e^{-\pi i\vartheta}H_{2,1}(xe^{-\pi i\sigma}\cdot e^{\pi i})\Big\}$$

$$= \frac{1}{\pi}\sum_{k=0}^\infty G_k x^{-k-1}\sin\pi(\vartheta-\sigma(k+1)) - \frac{1}{\pi\sigma}\sum_{k=0}^\infty (-)^k G'_k x^{-k_s}\sin\pi(\vartheta+\kappa k_s)$$

$$= \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-x)^{-k-1}\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{3}{2}+\sigma(k+1))} + \frac{1}{\sigma}\sum_{k=0}^\infty \frac{x^{-k_s}}{k!\Gamma(\frac{3}{2}-k_s)}. \tag{4.7}$$

In (4.2) it was necessary to assume that $\Gamma(\sigma n-\nu-\frac{1}{2})$ is regular for $n=0,1,2,\ldots$. If this assumption is false, the second expansion appearing in (4.7) is still valid; compare [12, §4]. The exponential component is from (4.5) given by

$$\hat{E}_{2,1}(x) \equiv \frac{i}{2\pi}\Big\{e^{\pi i\vartheta}E_{2,1}(xe^{\pi i\sigma}) - e^{-\pi i\vartheta}E_{2,1}(xe^{-\pi i\sigma})\Big\}$$

$$= \frac{A_0}{\pi}X^\vartheta e^{X\cos(\pi\sigma/\kappa)}\sum_{j=0}^\infty (-)^{j-1}c_j X^{-j}\sin\Big[X\sin\frac{\pi\sigma}{\kappa}+\frac{\pi}{\kappa}(\vartheta-j)\Big], \tag{4.8}$$

where $X=\kappa(hx)^{1/\kappa}$.

Let us denote the points $xe^{\pm\pi i\sigma}$ in the ζ-plane that appear in the arguments of the associated function ${}_2\Psi_1(\zeta)$ in (4.1) by $P_\pm$. Then when $0<\sigma<\frac{1}{3}$, $P_\pm$ lie in the exponentially large sector $|\arg\zeta|<\frac{1}{2}\pi\kappa$ in Fig. 2(a) and consequently $\hat{E}_{2,1}(x)$ is exponentially large as $x\to+\infty$. When $\sigma=\frac{1}{3}$, $P_\pm$ lie on the anti-Stokes lines $\arg\zeta = \pm\frac{1}{2}\pi\kappa$; on these rays $\cos\pi\sigma/\kappa = 0$ and $\hat{E}_{2,1}(x)$ is oscillatory with an algebraically controlled amplitude. When $\frac{1}{3} < \sigma < \frac{1}{2}$, $P_\pm$ lie in the exponentially small and algebraic sectors so that $\hat{E}_{2,1}(x)$ is exponentially small. When $\sigma = \frac{1}{2}$, $P_\pm$ lie on the Stokes lines $\arg\zeta = \pm\pi\kappa$, where the subdominant exponential expansions are in the process of switching off. Finally, when $\frac{1}{2} < \sigma < 1$, $P_\pm$ are situated in the algebraic sectors, where the expansions are purely algebraic.

Then we obtain the following theorem.

Theorem 4. *When $a = -\sigma$, with $0 < \sigma < 1$, we have the expansion of the generalised Struve function*

$$(\tfrac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\sigma) \sim \begin{cases} \hat{E}_{2,1}(x) + \hat{H}_{2,1}(x) & (0<\sigma<\frac{1}{2}) \\ \\ \hat{H}_{2,1}(x) & (\frac{1}{2}<\sigma<1) \end{cases} \tag{4.9}$$

as $z \to +\infty$, where $\hat{E}_{2,1}(x)$ and $\hat{H}_{2,1}(x)$ are defined in (4.5) and (4.6) with $x = z^2/4$, $X = \kappa(hx)^{1/\kappa}$. The quantities κ, h and A_0 are defined in (4.3) with $\vartheta = -\nu - \frac{3}{2}$. The first few coefficients $c_j = c_j(-\sigma, \nu)$ are given in (3.2).

When $\sigma = \frac{1}{2}$, the exponential expansion $\hat{E}_{2,1}(x)$ is maximally subdominant and is in the process of switching off. Thus, if we neglect this exponentially small contribution, the expansion of $(\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\frac{1}{2})$ as $z \to +\infty$ is given by $\hat{H}_{2,1}(x)$.

4.2. The expansion for $z \to i\infty$

When $z = i|z|$, we have upon replacing x by $xe^{\pi i}$ (so that $x = |z|^2/4$) and using the result ${}_2\Psi_1(\zeta e^{2\pi i}) = {}_2\Psi_1(\zeta)$

$$(\tfrac{1}{2}i|z|)^{-\nu-1}\mathbf{L}_\nu(i|z|;-\sigma) = \frac{i}{2\pi}\Big\{e^{\pi i\vartheta}{}_2\Psi_1(xe^{-\pi i\kappa}) - e^{-\pi i\vartheta}{}_2\Psi_1(xe^{\pi i\kappa})\Big\}. \tag{4.10}$$

Then, for $x > 0$, the associated functions ${}_2\Psi_1(xe^{\pm\pi i\kappa})$ have arguments situated on the Stokes lines, where the exponential contribution is maximally subdominant; see Fig. 2(b).

The algebraic contribution in (4.10) is

$$\begin{aligned} &\frac{i}{2\pi}\Big\{e^{\pi i\vartheta}H_{2,1}(xe^{-\pi i\kappa}\cdot e^{\pi i}) - e^{-\pi i\vartheta}H_{2,1}(xe^{\pi i\kappa}\cdot e^{-\pi i})\Big\} \\ = \; & -\frac{1}{\pi}\sum_{k=0}^\infty (-)^k G_k x^{-k-1}\sin\pi(\vartheta-\sigma(k+1)) - \frac{1}{\pi\sigma}\sum_{k=0}^\infty (-)^k G'_k x^{-k_s}\sin\pi(\vartheta-\sigma k_s) \\ = \; & \frac{1}{\pi}\sum_{k=0}^\infty \frac{\Gamma(k+\frac{1}{2})x^{-k-1}}{\Gamma(\nu+\frac{3}{2}+\sigma(k+1))}, \end{aligned}$$

the second sum vanishing since $\sin\pi(\vartheta-\sigma k_s) = \sin\pi k \equiv 0$ for $k = 0, 1, 2, \ldots$.

Then we have

Theorem 5. *When $a = -\sigma$, with $0 < \sigma < 1$, we have the expansion of the generalised Struve function*

$$(\tfrac{1}{2}i|z|)^{-\nu-1}\mathbf{L}_\nu(i|z|;-\sigma) \sim \frac{1}{\pi}\sum_{k=0}^\infty \frac{\Gamma(k+\frac{1}{2})(\frac{1}{2}|z|)^{-2k-2}}{\Gamma(\nu+\frac{3}{2}+\sigma(k+1))} \tag{4.11}$$

as $|z| \to \infty$. Here we have neglected the maximally subdominant exponentially small contribution to the expansion.

5. Numerical results Some numerical examples are presented to illustrate the expansions developed in Sections 3 and 4. We first consider the expansions valid when $a > 0$. In Table 1 we show the normalised coefficients c_j corresponding to $a = \frac{1}{2}$ and $\nu = \frac{1}{4}$ computed by means of the algorithm described in [8, Appendix A]. Table 2 shows values[5] of the generalised Struve function when $a > 0$. The top half of the table gives values of

$$\mathcal{L}_\nu(z;a) \equiv (\tfrac{1}{2}z)^{\nu-1}\mathbf{L}_\nu(z;a)$$

for different $z > 0$ when $a = \frac{1}{2}$, $\nu = \frac{1}{4}$ compared with the exponential expansion $E_{1,2}(\zeta)$ in (3.4). In the computation of $E_{1.2}(\zeta)$ we have employed the truncation index $j = 10$. The lower half of the table gives values of $\mathcal{L}_\nu(i|z|;a)$ with the dominant, optimally truncated[6] algebraic expansion $H_{1,2}(x)$ (with $x = |z|^2/4$) subtracted off. This value is compared with the exponential expansion $\hat{E}_1(X)$ defined in (3.5).

Table 1: The normalised coefficients $c_j = A_j/A_0$ for ${}_1\Psi_2(\zeta)$ in (3.1) when $a = \frac{1}{2}$ and $\nu = \frac{1}{4}$.

j	c_j	j	c_j
1	$-\frac{5}{12}$	2	$-\frac{35}{288}$
3	$-\frac{665}{10368}$	4	$+\frac{9625}{497664}$
5	$+\frac{1856855}{5971968}$	6	$+\frac{606631025}{429981696}$
7	$+\frac{27773871125}{5159780352}$	8	$+\frac{8996211899675}{495338913792}$
9	$+\frac{2459153764892825}{53496602689536}$	10	$-\frac{22173972436540925}{1283918464548864}$

We now consider the expansions valid when $-1 < a < 0$. Table 3 shows values of $\mathcal{L}_\nu(z;-\sigma)$ for different $z > 0$ and σ in the range $0 < \sigma < 1$ compared

[5]In the tables we write $x(y)$ to represent $x \times 10^y$.

[6]That is, truncated at, or near, the term of smallest magnitude.

Table 2: Values of $\mathcal{L}_\nu(z;a) \equiv (\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;a)$ compared with the asymptotic expansions in (3.4) and (3.5) when $a = \frac{1}{2}$ and $\nu = \frac{1}{4}$. The exponential expansions have truncation index $j \leq 10$ and the algebraic expansion $H_{1,2}(x)$ (with $x = |z|^2/4$) has been optimally truncated.

z	$\mathcal{L}_\nu(z;a)$	$E_{1,2}(\zeta)$
5	$+3.452097942\,(1)$	$+3.461544352\,(1)$
10	$+1.226039040\,(5)$	$+1.226039286\,(5)$
12	$+6.877617187\,(6)$	$+6.877617204\,(6)$
15	$+5.182624938\,(9)$	$+5.182624938\,(9)$
z	$\mathcal{L}_\nu(z;a) - H_{1,2}(x)$	$\hat{E}_1(X)$
$10i$	$-4.572292174\,(-6)$	$-4.571953242\,(-6)$
$15i$	$+4.021530098\,(-10)$	$+4.021543491\,(-10)$
$20i$	$+6.827666326\,(-12)$	$+6.827666325\,(-12)$
$25i$	$+3.515426867\,(-15)$	$+3.515426867\,(-15)$

Table 3: Values of $\mathcal{L}_\nu(z;-\sigma) \equiv (\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\sigma)$ compared with the asymptotic expansions in (4.9) when $z > 0$ and $\nu = \frac{1}{3}$. The exponential expansion has been optimally truncated.

	$\sigma = 1/5$			$\sigma = 1/3$	
z	$\mathcal{L}_\nu(z;-\sigma)$	Asymptotic	z	$\mathcal{L}_\nu(z;-\sigma)$	Asymptotic
8	$+1.371278215\,(4)$	$+1.371994397\,(4)$	5	$+4.994707877\,(1)$	$+4.992261627\,(1)$
10	$-1.628234940\,(7)$	$-1.628235076\,(7)$	8	$+5.127188845\,(2)$	$+5.127188845\,(2)$
15	$-2.287676991\,(22)$	$-2.287676991\,(22)$	10	$+1.563077837\,(3)$	$+1.563077837\,(3)$
	$\sigma = 1/2$			$\sigma = 3/5$	
z	$\mathcal{L}_\nu(z;-\sigma)$	Asymptotic	z	$\mathcal{L}_\nu(z;-\sigma)$	Asymptotic
3	$+4.705453951\,(0)$	$+4.719691159\,(0)$	3	$+4.075339511\,(0)$	$+4.074935642\,(0)$
5	$+1.918197617\,(1)$	$+1.918197638\,(1)$	4	$+7.439302510\,(0)$	$+7.439299037\,(0)$
8	$+8.747082153\,(1)$	$+8.747082153\,(1)$	5	$+1.276299496\,(1)$	$+1.276299496\,(1)$

with the asymptotic expansion in (4.9). For each value of σ the normalised coefficients c_j were computed. The expansions have been optimally truncated and we have used the result $\mathcal{L}_\nu(z;-\sigma) \sim H_{2,1}(x)$ as $z \to +\infty$ (with $x = z^2/4$) when $\sigma = \frac{1}{2}$. Finally, in Table 4 we show values of $\mathcal{L}_\nu(i|z|;-\sigma)$ for different $|z|$ and σ in the range $0 < \sigma < 1$ compared with the optimally truncated algebraic expansion in (4.11).

It is seen in all cases that very good agreement is achieved when z is sufficiently large.

Table 4: Values of $\mathcal{L}_\nu(z;-\sigma)\equiv(\frac{1}{2}z)^{-\nu-1}\mathbf{L}_\nu(z;-\sigma)$ compared with the asymptotic expansion (4.11) when $\arg z=\frac{1}{2}\pi$ and $\nu=\frac{4}{3}$. The algebraic expansion (4.11) has been optimally truncated.

z	$\sigma=1/4$ $\mathcal{L}_\nu(z;-\sigma)$	Asymptotic	z	$\sigma=1/3$ $\mathcal{L}_\nu(z;-\sigma)$	Asymptotic
$6i$	3.044656205 (−2)	3.044653596 (−2)	$6i$	2.792844201 (−2)	2.792844405 (−2)
$8i$	1.673275565 (−2)	1.673275565 (−2)	$8i$	1.539185802 (−2)	1.539185802 (−2)
z	$\sigma=1/2$ $\mathcal{L}_\nu(z;-\sigma)$	Asymptotic	z	$\sigma=3/4$ $\mathcal{L}_\nu(z;-\sigma)$	Asymptotic
$4i$	5.552864403 (−2)	5.553062223 (−2)	$3i$	7.704243224 (−2)	7.704358006 (−2)
$5i$	3.420993477 (−2)	3.420993479 (−2)	$4i$	4.087728092 (−2)	4.087728092 (−2)

Appendix: The coefficients $c_j\equiv c_j(-\sigma,\nu)$ in the expansion (4.5) The inverse factorial expansion (2.6) involving the coefficients $c_j\equiv c_j(\alpha,\nu)$ for the function ${}_1\Psi_2$ in (3.1) can be written in the form (see [8, Appendix A])

$$R(\alpha)\equiv\frac{\Gamma(\kappa s+\vartheta')}{\Gamma(\alpha s+\nu+\frac{3}{2})\Gamma(s+\frac{3}{2})}=\kappa A_0(h\kappa^\kappa)^s\biggl\{\sum_{j=0}^{M-1}\frac{c_j(\alpha,\nu)}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa s+\vartheta')_M}\biggr\}$$

as $|s|\to\infty$ in $|\arg s|<\pi$, where $\kappa=1+\alpha$, $h=\alpha^{-\alpha}$, $\vartheta'=-\nu-\frac{3}{2}$ and $A_0=(\kappa/\alpha)^{\nu+1}/\sqrt{2\pi}$. Then, when $\alpha=-\sigma$, $0<\sigma<1$, we have

$$R(-\sigma)=\frac{\kappa}{\sqrt{2\pi}}\Bigl(\frac{\kappa}{-\sigma}\Bigr)^{\nu+1}((-\sigma)^\sigma\kappa^\kappa)^s\biggl\{\sum_{j=0}^{M-1}\frac{c_j(-\sigma,\nu)}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa s+\vartheta')_M}\biggr\}$$

$$=\frac{\kappa}{\sqrt{2\pi}}\Bigl(\frac{\kappa}{\sigma}\Bigr)^{\nu+1}(\sigma^\sigma\kappa^\kappa)^s e^{\pi i(\sigma s-\nu-1)}\biggl\{\sum_{j=0}^{M-1}\frac{c_j(-\sigma,\nu)}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa a+\vartheta')_M}\biggr\}.$$

The inverse factorial expansion (2.6) for the function ${}_2\Psi_1$ in (4.2) becomes (with κ, h and A_0 defined in (4.3))

$$S\equiv\frac{\Gamma(\kappa s+\vartheta')\Gamma(\sigma s-\nu-\frac{1}{2})}{\Gamma(s+\frac{3}{2})}$$

$$=\kappa\sqrt{2\pi}\Bigl(\frac{\kappa}{\sigma}\Bigr)^{\nu+1}(\sigma^\sigma\kappa^\kappa)^s\biggl\{\sum_{j=0}^{M-1}\frac{d_j}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa s+\vartheta')_M}\biggr\},\qquad\text{(A.1)}$$

where d_j are coefficients to be determined. But

$$S=\frac{\pi R(-\sigma)}{\sin\pi(\sigma s-\nu-\frac{1}{2})}=\frac{2\pi R(-\sigma)}{e^{\pi i(\sigma s-\nu-1)}\Lambda(s)}$$

$$= \frac{\kappa\sqrt{2\pi}}{\Lambda(s)} \left(\frac{\kappa}{\sigma}\right)^{\nu+1} (\sigma^\sigma \kappa^\kappa)^s \biggl\{ \sum_{j=0}^{M-1} \frac{c_j(-\sigma,\nu)}{(\kappa s+\vartheta')_j} + \frac{O(1)}{(\kappa a+\vartheta')_M} \biggr\}, \tag{A.2}$$

where $\Lambda(s) = 1 - e^{-2\pi i(\sigma s-\nu-\frac{1}{2})}$.

Letting $|s| \to \infty$ in the sector $-\pi < \arg s < 0$, so that $\Lambda(s) \to 1$, we have upon comparison of the terms in (A.1) and (A.2) that $d_j = c_j(-\sigma,\nu)$ $(1 \leq j \leq M-1)$.

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