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sup $\times$ inf inequality on manifold of dimension 3

Samy Skander Bahoura

Department of Mathematics, Patras University, 26500 Patras , Greece
samybahoura@yahoo.fr

Abstract

We give an estimate of type $\sup \times \inf$ on riemannian manifold of dimension 3 for the prescribed curvature equation.

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1 Introduction and Main Results

In dimension 3, the scalar curvature equation is:

$$8\Delta u + R_g u = V u^5, \;\; u > 0. \qquad (E)$$

Where R_g is the scalar curvature and V is a function (called the prescribed scalar curvature).

We consider three positive real number a, b, A and we suppose V lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } ||\nabla V||_{L^\infty(M)} \leq A. \qquad (C)$$

The equation (E) was studied a lot when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example [2], [6], [9]. In these cases we have some inequalities of type $\sup \times \inf$.

The corresponding equation in dimension 2, on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u = V e^u, \qquad (E')$$

The equation (E') was studed a lot and we can find many important results about a priori estimates in [3], [4], [7], [10], and [13].

In the case $V \equiv 1$ and M compact, the equation (E) is Yamabe equation. T.Aubin and R.Schoen have proved the existence of solution in this case, see for example [1] and [8].

When M is a compact riemannian manifold, there is some compactness results for the equation (E) see [11-12]. Li and Zhu, see [12], proved that the energy is bounded, and, if we assume M not diffeormorfic to the three sphere, the solutions are uniformly bounded. They use the positive mass theorem.

Now, if we suppose M a riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [11] proved that the product $\sup \times \inf$ is bounded.

Here, we give an equality of type $\sup \times \inf$ for the equation (E) with general conditions (C). We have:

Theorem 1.1 *For all compact set K of M and all positive numbers a,b,A, there is a positive constant c, which depends only on, a,b,A,K,M,g such that:*

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (E) with conditions (C).

As a consequence of the previous theorem, we have an estimate of the maximum if we control the minimum of the solutions:

Corollary 1.2 *For all compact set K of M and all positive numbers a,b,A, m, there is a positive constant c, which depends only on, a,b,A, m, K,M,g such that:*

$$\sup_K u \leq c, \text{ if } \inf_M u \geq m > 0,$$

for all u solution of (E) with conditions (C).

Note that in our work, we have not assumption on energy or boundary condition if we assume the manifold M with boundary.

Next, in the proof of the previous theorem, we can replace the scalar curvature by any smooth function f, but here we proof the result with R_g the scalar curvature.

2 Proof of the Theorem

Part I: The metric and the laplacian in polar coordinates.

Let (M, g) a Riemannian manifold. We note $g_{x,ij}$ the local expression of the metric g in the exponential map centered in x.

We are concerning by the polar coordinates expression of the metric. By using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{ij}^k(r,\theta)d\theta^i d\theta^j = dt^2 + r^2\tilde{g}_{ij}^k(r,\theta)d\theta^i d\theta^j = g_{x,ij}dx^i dx^j,$$

in a polar chart with origin x", $]0,\epsilon_0[\times U^k$, with (U^k,ψ) a chart of $\mathbb{S}_{n-1}$. We can write the element volume:

$$dV_g = r^{n-1}\sqrt{|\tilde{g}^k|}drd\theta^1\ldots d\theta^{n-1} = \sqrt{[det(g_{x,ij})]}dx^1\ldots dx^n,$$

then,

$$dV_g = r^{n-1}\sqrt{[det(g_{x,ij})]}[\exp_x(r\theta)]\alpha^k(\theta)drd\theta^1\ldots d\theta^{n-1},$$

where, α^k is such that, $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta)d\theta^1\ldots d\theta^{n-1}$. (Riemannian volume element of the la sphere in the chart (U^k,ψ)).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta)\sqrt{[det(g_{x,ij})]},$$

Clearly, we have the following proposition:

Propostion 2.1 *Let $x_0\in M$, there exist $\epsilon_1>0$ and if we reduce U^k, we have:*

$$|\partial_r\tilde{g}_{ij}^k(x,r,\theta)|+|\partial_r\partial_{\theta^m}\tilde{g}_{ij}^k(x,r,\theta)| \leq Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

and,

$$|\partial_r|\tilde{g}^k|(x,r,\theta)|+\partial_r\partial_{\theta^m}|\tilde{g}^k|(x,r,\theta) \leq Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

Remark:

$\partial_r[\log\sqrt{|\tilde{g}^k|}]$ is a local function of θ, and the restriction of the global function on the sphere $\mathbb{S}_{n-1}$, $\partial_r[\log\sqrt{det(g_{x,ij})}]$. We will note, $J(x,r,\theta) = \sqrt{det(g_{x,ij})}$.

Let's write the laplacian in $[0,\epsilon_1]\times U^k$,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log\sqrt{|\tilde{g}^k|}]\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r \log J(x,r,\theta)\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We write the laplacian (radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log J(x,r,\theta)]\partial_r - \Delta_{S_r(x)},$$

where $\Delta_{S_r(x)}$ is the laplacian on the sphere $S_r(x)$.

We set $L_\theta(x,r)(...) = r^2\Delta_{S_r(x)}(...)[\exp_x(r\theta)]$, clearly, this operator is a laplacian on $\mathbb{S}_{n-1}$ for particular metric. We write,

$$L_\theta(x,r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[J(x,r,\theta)]\partial_r - \frac{1}{r^2}L_\theta(x,r).$$

If, u is function on M, then, $\bar{u}(r,\theta) = u[\exp_x(r\theta)]$ is the corresponding function in polar coordinates centered in x. We have,

$$-\Delta u = \partial_{rr}\bar{u} + \frac{n-1}{r}\partial_r\bar{u} + \partial_r[J(x,r,\theta)]\partial_r\bar{u} - \frac{1}{r^2}L_\theta(x,r)\bar{u}.$$

Part II: "Blow-up" and "Moving-plane" methods

The "blow-up" technique

Let, $(u_i)_i$ a sequence of functions on M such that,

$$8\Delta u_i + R_g u_i = V_i {u_i}^5, \;\; u_i > 0, \qquad (E)$$

We argue by contradiction and we suppose that $\sup^{1/3} \times \inf$ is not bounded.

We assume that:

$\forall\ c, R > 0\ \exists\ u_{c,R}$ solution of (E) such that:

$$R[\sup_{B(x_0,R)} u_{c,R}]^{1/3} \times \inf_M u_{c,R} \geq c, \qquad (H)$$

Propostion 2.2 *There exist a sequence of points $(y_i)_i$, $y_i \to x_0$ and two sequences of positive real number $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we consider $v_i(y) = \dfrac{u_i[\exp_{y_i}(y)]}{u_i(y_i)}$, we have:*

$$0 < v_i(y) \leq \beta_i \leq 2^{1/2}, \ \beta_i \to 1.$$

$$v_i(y) \to \left(\frac{1}{1+|y|^2} \right)^{1/2}, \text{ uniformly on every compact set of } \mathbb{R}^3.$$

$$l_i[u_i(y_i)]^{1/3} \times \inf_M u_i \to +\infty$$

Proof:

We use the hypothesis (H), we can take two sequences $R_i > 0, R_i \to 0$ and $c_i \to +\infty$, such that,

$$R_i[\sup_{B(x_0,R_i)} u_i]^{1/3} \times \inf_M u_i \geq c_i \to +\infty,$$

Let, $x_i \in B(x_0, R_i)$, such that $\sup_{B(x_0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x, x_i)]^{1/2} u_i(x), x \in B(x_i, R_i)$. Then, $x_i \to x_0$.

We have,

$$\max_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = {R_i}^{1/2} u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$

Set :

$$l_i = R_i - d(y_i, x_i), \ \bar{u}_i(y) = u_i[\exp_{y_i}(y)], \ v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)]^2)]}{u_i(y_i)}.$$

Clearly, $y_i \to x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2}}[u_i(y_i)]^2 = \frac{[s_i(y_i)]^2}{c_i^{1/2}} \geq \frac{c_i^1}{c_i^{1/2}} = c_i^{1/2} \to +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]^2] \in B(y_i, \delta_i l_i)$ with $\delta_i = \dfrac{1}{(c_i)^{1/2}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$, we can write,

$$u_i(y)[R_i - d(y, y_i)]^{1/2} \leq u_i(y_i)(l_i)^{1/2}.$$

But, $d(y,y_i)\leq \delta_i l_i$, $R_i>l_i$ and $R_i-d(y,y_i)\geq R_i-\delta_i l_i>l_i-\delta_i l_i=l_i(1-\delta_i)$, we obtain,

$$0<v_i(z)=\frac{u_i(y)}{u_i(y_i)}\leq \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{1/2}\leq 2^{1/2}.$$

We set, $\beta_i=\left(\dfrac{1}{1-\delta_i}\right)^{1/2}$, clearly $\beta_i\to 1$.

The function v_i is solution of:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk}v_i-\partial_k\left[g^{jk}\sqrt{|g|}\right][\exp_{y_i}(y)]\partial_j v_i+\frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^4}v_i=V_i{v_i}^5,$$

By elleptic estimates and Ascoli, Ladyzenskaya theorems, $(v_i)_i$ converge uniformely on each compact to the function v solution on $\mathbb{R}^3$ of,

$$8\Delta v=V(x_0)v^5,\ v(0)=1,\ 0\leq v\leq 1\leq 2^{1/2},$$

Without loss of generality, we can suppose $V(x_0)=24$.

By using maximum principle, we have $v>0$ on $\mathbb{R}^3$, the result of Caffarelli-Gidas-Spruck (see [5]) give, $v(y)=\left(\dfrac{1}{1+|y|^2}\right)^{1/2}$. We have the same propreties for v_i in the previous paper [2].

Polar coordinates and "moving-plane" method

Let,

$$w_i(t,\theta)=e^{1/2}\bar u_i(e^t,\theta)=e^{t/2}u_i[\exp_{y_i}(e^t\theta)],\ \text{ et } a(y_i,t,\theta)=\log J(y_i,e^t,\theta).$$

Lemma 2.3 *The function w_i is solution of:*

$$-\partial_{tt}w_i-\partial_t a\partial_t w_i-L_\theta(y_i,e^t)+cw_i=V_iw_i^5,$$

with,

$$c=c(y_i,t,\theta)=\left(\frac{1}{2}\right)^2+\frac{1}{2}\partial_t a-\lambda e^{2t},$$

Proof:

We write:

$$\partial_t w_i = e^{3t/2}\partial_r \bar u_i + \frac{1}{2} w_i,\ \ \partial_{tt} w_i = e^{5t/2}\left[\partial_{rr}\bar u_i + \frac{2}{e^t}\partial_r \bar u_i\right] + \left(\frac{1}{2}\right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \partial_t a \partial_t w_i = e^{5t/2}\left[\partial_r \log J \partial_r \bar u_i\right] + \frac{1}{2}\partial_t a w_i.$$

the lemma is proved.

Now we have, $\partial_t a = \dfrac{\partial_t b_1}{b_1}$, $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$,

We can write,

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t,\theta)] w_i = V_i {w_i}^{N-1},$$

where, $b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \dfrac{1}{2\sqrt{b_1}}\partial_{tt} b_1 - \dfrac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2.$

Let,

$$\tilde w_i = \sqrt{b_1} w_i.$$

Lemma 2.4 *The function $\tilde w_i$ is solution of:*

$$-\partial_{tt}\tilde w_i + \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\tilde w_i) + 2\nabla_\theta(\tilde w_i).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)\tilde w_i =$$

$$= V_i \left(\frac{1}{b_1}\right)^2 \tilde w_i^5,$$

where, $c_2 = [\dfrac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2].$

<u>**Proof:**</u>

We have:

$$-\partial_{tt}\tilde w_i - \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}} w_i + (c + b_2)\tilde w_i = V_i \left(\frac{1}{b_1}\right)^2 \tilde w_i^5,$$

But,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1}w_i)=\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}w_i-2\nabla_\theta w_i.\nabla_\theta\sqrt{b_1}+w_i\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i)=w_i\nabla_\theta\sqrt{b_1}+\sqrt{b_1}\nabla_\theta w_i,$$

we deduce,

$$\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}w_i=\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\tilde w_i)+2\nabla_\theta(\tilde w_i).\nabla_\theta\log(\sqrt{b_1})-c_2\tilde w_i,$$

with $c_2=[\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1})+|\nabla_\theta\log(\sqrt{b_1})|^2]$. The lemma is proved.

The "moving-plane" method:

Let ξ_i a real number, and suppose $\xi_i\leq t$, we set $t^{\xi_i}=2\xi_i-t$ and $\tilde w_i^{\xi_i}(t,\theta)=\tilde w_i(t^{\xi_i},\theta)$.

We have,

$$-\partial_{tt}\tilde w_i^{\xi_i}+\Delta_{g_{y_i,e^{t^{\xi_i}}\mathbb{S}_2}}(\tilde w_i)+2\nabla_\theta(\tilde w_i^{\xi_i}).\nabla_\theta\log(\sqrt{b_1})\tilde w_i^{\xi_i}+[c(t^{\xi_i})+b_1^{-1/2}(t^{\xi_i},.)b_2(t^{\xi_i})-c_2^{\xi_i}]\tilde w_i^{\xi_i}=$$

$$=V_i^{\xi_i}\left(\frac{1}{b_1^{\xi_i}}\right)^2(\tilde w_i^{\xi_i})^5.$$

By using the same arguments than in [2], we have:

Propostion 2.5

$$1)\ \tilde w_i(\lambda_i,\theta)-\tilde w_i(\lambda_i+4,\theta)\geq\tilde k>0,\ \forall\ \theta\in\mathbb{S}_2.$$

For all $\beta>0$, there exist $c_\beta>0$ such than:

$$2)\ \frac{1}{c_\beta}e^{t/2}\leq\tilde w_i(\lambda_i+t,\theta)\leq c_\beta e^{t/2},\ \forall\ t\leq\beta,\ \forall\ \theta\in\mathbb{S}_2.$$

We set,

$$\bar Z_i=-\partial_{tt}(...)+\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(...)+2\nabla_\theta(...).\nabla_\theta\log(\sqrt{b_1})+(c+b_1^{-1/2}b_2-c_2)(...)$$

Remark: In the opertor $\bar{Z}_i$, by using the proposition 3, the coeficient $c + b_1^{-1/2} b_2 - c_2$ satisfy:

$$c + b_1^{-1/2} b_2 - c_2 \geq k' > 0, \text{ for } t << 0,$$

it is fundamental if we want to apply Hopf maximum principle.

Goal:

Like in [2], we have elliptic second order operator, here it's $\bar{Z}_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0.$$

We write:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) = (\Delta_{g_{y_i, e^{t\xi_i}, \mathbb{S}_2}} - \Delta_{g_{y_i, e^t, \mathbb{S}_2}})(\tilde{w}_i^{\xi_i}) +$$

$$+2(\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t})(w_i^{\xi_i}).\nabla_{\theta, e^{t\xi_i}} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta, e^t}(\tilde{w}_i^{\xi_i}).\nabla_{\theta, e^{t\xi_i}}[\log(\sqrt{b_1^{\xi_i}}) - \log\sqrt{b_1}] +$$

$$+2\nabla_{\theta, e^t} w_i^{\xi_i}.(\nabla_{\theta, e^{t\xi_i}} - \nabla_{\theta, e^t}) \log\sqrt{b_1} - [(c + b_1^{-1/2} b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2} b_2 - c_2)]\tilde{w}_i^{\xi_i} +$$

$$+V_i^{\xi_i}\left(\frac{1}{b_1^{\xi_i}}\right)^2 (\tilde{w}_i^{\xi_i})^5 - V_i\left(\frac{1}{b_1}\right)^2 \tilde{w}_i^5. \qquad (***1)$$

Clearly, we have:

Lemma 2.6

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} Ricci_{y_i}(\theta, \theta) e^{2t} + \ldots,$$

$$R_g(e^t\theta) = R_g(y_i) + < \nabla R_g(y_i)|\theta > e^t + \ldots.$$

According to proposition 2.1 and lemma 2.6, we have

Propostion 2.7

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq V_i {b_1}^{(-2)}[(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + 2(\tilde{w}_i^{\xi_i})^5 |V_i^{\xi_i} - V_i| +$$

$$+C|e^{2t} - e^{2t\xi_i}| \left[|\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})| + |Ricci_{y_i}|[\tilde{w}_i^{\xi_i} + (\tilde{w}_i^{\xi_i})^5] + |R_g(y_i)|\tilde{w}_i^{\xi_i}\right] + C'|e^{3t\xi_i} - e^{3t}|.$$

Proof:

We use proposition 2.1, we have:

$$a(y_i,t,\theta)=\log J(y_i,e^t,\theta)=\log b_1, |\partial_t b_1(t)|+|\partial_{tt}b_1(t)|+|\partial_{tt}a(t)|\leq Ce^{2t},$$

and,

$$|\partial_{\theta_j}b_1|+|\partial_{\theta_j,\theta_k}b_1|+\partial_{t,\theta_j}b_1|+|\partial_{t,\theta_j,\theta_k}b_1|\leq Ce^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i})-\partial_t b_1(t)|\leq C'|e^{2t}-e^{2t^{\xi_i}}|, \text{ on }]-\infty,\log\epsilon_1]\times\mathbb{S}_2, \forall\ x\in B(x_0,\epsilon_1)$$

Locally,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_2}}=L_\theta(y_i,e^t)=-\frac{1}{\sqrt{|\tilde g^k(e^t,\theta)|}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k(e^t,\theta)|}\partial_{\theta^j}].$$

Thus, in $[0,\epsilon_1]\times U^k$, we have,

$$A_i=\left[\left[\frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right]^{\xi_i}-\frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right](\tilde w_i^{\xi_i})$$

then, $A_i=B_i+D_i$ with,

$$B_i=\left[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta)-\tilde g^{\theta^l\theta^j}(e^t,\theta)\right]\partial_{\theta^l\theta^j}\tilde w_i^{\xi_i}(t,\theta),$$

and,

$$D_i=\left[\frac{1}{\sqrt{|\tilde g^k|(e^{t^{\xi_i}},\theta)}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta)\sqrt{|\tilde g^k|}(e^{t^{\xi_i}},\theta)]-\frac{1}{\sqrt{|\tilde g^k|(e^t,\theta)}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k|}(e^t,\theta)]\right]\partial_{\theta^j}\tilde w_i^{\xi_i}(t$$

we deduce,

$$A_i\leq C_k|e^{2t}-e^{2t^{\xi_i}}|\left[|\nabla_\theta\tilde w_i^{\xi_i}|+|\nabla_\theta^2(\tilde w_i^{\xi_i})|\right],$$

We take $C=\max\{C_i, 1\leq i\leq q\}$ and if we use $(***1)$, we obtain proposition 2.7.

We have,

$$c(y_i,t,\theta)=\left(\frac{1}{4}\right)+\frac{1}{2}\partial_t a+R_g e^{2t}, \qquad (\alpha_1)$$

$$b_2(t,\theta)=\partial_{tt}(\sqrt{b_1})=\frac{1}{2\sqrt{b_1}}\partial_{tt}b_1-\frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2, \qquad (\alpha_2)$$

$$c_2=[\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1})+|\nabla_\theta \log(\sqrt{b_1})|^2], \qquad (\alpha_3)$$

Then,

$$\partial_t c(y_i,t,\theta)=\frac{1}{2}\partial_{tt}a+2e^{2t}R_g(e^t\theta)+e^{3t}<\nabla R_g(e^t\theta)|\theta>,$$

by proposition 1,

$$|\partial_t c_2|+|\partial_t b_1|+|\partial_t b_2|+|\partial_t c|\leq K_1 e^{2t},$$

Now, we consider the function, $\bar{w}_i(t,\theta)=\tilde{w}_i(t,\theta)-\dfrac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}e^t$, and $\lambda>2>0$.

For $t\leq t_i=-(2/3)\log u_i(y_i)$, we have:

$$\bar{w}_i(t,\theta)=e^t\left[b_1(t,\theta)e^{-t/2}u_i o \exp_{y_i}(e^t\theta)-\frac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}\right]\geq$$

$$\geq e^t\frac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}>0,$$

We set, $\mu_i=\dfrac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}$.

We use proposition 2.5 and the same arguments than in [2], we obtain:

Lemma 2.8 *There exists $\nu<0$ such that for $\mu\leq\nu$:*

$$\bar{w}_i^\mu(t,\theta)-\bar{w}_i(t,\theta)\leq 0,\ \forall\ (t,\theta)\in[\mu,t_i]\times\mathbb{S}_2,$$

We set, $\lambda_i=-2\log u_i(y_i)$, then,

Lemma 2.9

$$\bar{w}_i(\lambda_i,\theta)-\bar{w}_i(\lambda_i+4,\theta)>0.$$

Proof of lemma 2.9:

Clearly:

$$\bar{w}_i(\lambda_i, \theta) - \bar{w}_i(\lambda_i + 4, \theta) = \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) + \mu_i e^{\lambda_i}(e^4 - 1),$$

we deduce lemma 2.9 from proposition 2.5.

Let, $\xi_i = \sup\{\mu \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \ \forall \ (t, \theta) \in [\xi_i, t_i] \times \mathbb{S}_2\}$.

The real ξ_i exists (see [2]), if we use $(* * *2)$, we have:

$$\tilde{w}_i^{\xi_i}(t, \theta) + |\nabla_\theta \tilde{w}_i^{\xi_i}(t, \theta)| + |\nabla_\theta^2 \tilde{w}_i^{\xi_i}(t, \theta)| \leq C(R), \ \forall \ (t, \theta) \in]-\infty, \log R] \times \mathbb{S}_2,$$

We can write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) = \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) - \mu_i \bar{Z}_i(e^{t^{\xi_i}} - e^t),$$

$$-\bar{Z}_i(e^{t^{\xi_i}} - e^t) = [1 - \frac{1}{4} - \frac{3}{2}\partial_t a - R_g e^{2t} + b_1^{-1/2} b_2 - c_2](e^{t^{\xi_i}} - e^t) \leq c_1(e^{t^{\xi_i}} - e^t),$$

with $c_1 > 0$, because $|\partial_t a| + |\partial_t b_1| + |\partial_{tt} b_1| + |\partial_{t,\theta_j} b_1| + |\partial_{t,\theta_j,\theta_k} b_1| \leq C' e^{2t} < 1$, for t very small.

We use proposition 2.7, to obtain on, $[\xi_i, t_i] \times \mathbb{S}_2$,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq c_2' V_i[(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + |V_i^{\xi_i} - V_i|(w_i^{\xi_i})^5 + [\mu_i c_1 - C'(R)](e^{t^{\xi_i}} - e^t) \leq 0,$$

with $c_2' > 0$.

Like in [2], after using Hopf maximum principle, we have,

$$\sup_{\theta \in \mathbb{S}_2} [\bar{w}_i^{\xi_i}(t_i, \theta) - \bar{w}_i(t_i, \theta)] = 0.$$

We have:

$$\bar{w}_i^{\xi_i}(t_i, \theta_i) - \bar{w}_i(t_i, \theta_i) = 0, \ \forall \ i.$$

We deduce,

$$[u_(y_i)]^{1/3} \times \inf_M u_i \leq c, \ \forall \ i.$$

it is in contradiction with proposition 2.

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