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Structures of solvable 3-Lie algebras

BAI Ruipu

College of Mathematics and Computer Science,
Hebei University, Baoding, 071002, China
email: bairuipu@hbu.cn

LI Qiyong

College of Mathematics and Computer Science,
Hebei University, Baoding, 071002, China

Abstract

This paper considers structures of a class of solvable 3-Lie algebras which have a filiform 3-Lie algebra as a maximal Hypo-nilpotent ideal. It is proved that there does not exist metric structures on the 3-Lie algebras. And the concrete expression of derivations is given, and it is proved that there exist only two exterior derivations. The result can be used in the study of solvable 3-Lie algebras.

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1 Introduction

3-Lie algebra [1] has close relationships in many fields on mathematics, mathematical physics and string theory (cf. [2, 3, 4, 5]). In paper [6], a class of solvable 3-Lie algebras which have a filiform 3-Lie algebra as a maximal Hypo-nilpotent ideal was constructed, and the completely classification was given. In this paper we study the metric structures and derivation algebras of these 3-Lie algebras.

A 3-Lie algebra is a vector space L over a field F endowed with a 3-ary multi-linear skew-symmetric operation $[x_1, x_2, x_3]$ satisfying the 3-Jacobi identity

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3}[x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3], \ \forall x_1, x_2, x_3 \in L. \quad (1)$$

A derivation of a 3-Lie algebra L is a linear map $D : L \to L$, such that for any elements x_1, x_2, x_3 of L

$$D([x_1, x_2, x_3]) = \sum_{i=1}^{3} [x_1, \ \cdots, \ D(x_i), \ \cdots, \ x_3]. \tag{2}$$

The set of all derivations of L is a subalgebra of Lie algebra $\mathrm{gl}(L)$. This subalgebra is called *the derivation algebra* of A, and is denoted by $\mathrm{Der}L$. The map $\mathrm{ad}(x_1, x_2) : \ L \to \ L$ defined by $\mathrm{ad}(x_1, x_2)(x) = [x_1, x_2, x]$ for $x_1, x_2, \ x \in L$ is called *a left multiplication.* It follows from (2) that $\mathrm{ad}(x_1, x_2)$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $\mathrm{Der}L$ and is denoted by $\mathrm{ad}(L)$. Every element in $\mathrm{ad}(A)$ is by definition *an inner derivation*, and for $\forall \ x_1, x_2, y_1, y_2$ of L,

$$[\mathrm{ad}(x_1, x_2), \mathrm{ad}(y_1, y_2)] = \mathrm{ad}([x_1, x_2, y_1], y_2) + \mathrm{ad}(y_1, [x_1, x_2, y_2]).$$

An ideal of a 3-Lie algebra L is a subspace I such that $[I, L, L] \subseteq I$. An ideal I of a 3-Lie algebra L is called *a solvable ideal*, if $I^{(r)} = 0$ for some $r \geq 0$, where $I^{(0)} = I$ and $I^{(s)}$ is defined by induction, $I^{(s+1)} = [I^{(s)}, I^{(s)}, L]$ for $s \geq 0$. When $L = I$, L is *a solvable 3-Lie algebra.*

An ideal I of is called *a nilpotent ideal*, if I satisfies $I^r = 0$ for some $r \geq 0$, where $I^0 = I$ and $I^{r+1} = [I^r, I, L]$ for $r \geq 0$. If $I = L$, L is called *a nilpotent* 3-*Lie algebra.*

Let L be a 3-Lie algebra and I be an ideal of L. If I is a nilpotent subalgebra but is not a nilpotent ideal, then I is called *a hypo-nilpotent ideal* of L. If I is not properly contained in any hypo-nilpotent ideals, then I is called *a maximal hypo-nilpotent ideal* of L.

Let L be a nilpotent m-dimensional 3-Lie algebra over a field F. If the lower central series L^i $(i \geq 1)$ satisfy the following condition: $\dim L^i = m - (2+i), \ i \geq 1$, then L is called *a filiform 3-Lie algebra.* Denotes N the filiform 3-Lie algebra with the multiplication in a basis $e_1, \ \cdots, e_m$ as follows

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2}, 5 \leq j \leq m-1. \end{cases}$$

Lemma 1.1 *Let L be an $(m+k)$-dimensional solvable but non-nilpotent 3-Lie algebra with the maximal Hypo-nilpotent ideal N, where $k \geq 1, m \geq 5$, then we have $k = 1$. And there exists a basis $\{e_1, \cdots, e_m, e_{m+1}\}$, such that the multiplication of L is as follows:*

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2}, 5 \leq j \leq m-1, \\ [e_{m+1}, e_1, e_2] = e_2, \\ [e_{m+1}, e_1, e_k] = (m-k+2)e_k, 3 \leq k \leq m. \end{cases} \tag{3}$$

In the following, L is the 3-Lie algebra in Lemma 1.1, and the characteristic of the field F is zero.

2 Metric Structures on the 3-Lie Algebra L

In this section we discuss the metric properties of the 3-Lie algebra L.

A metric on the 3-Lie algebra L is a symmetric bilinear form $B : L \times L \to F$ satisfying

$$B([x, y, z], u) + B(z, [x, y, u]) = 0, \forall x, y, z, u \in L. \tag{4}$$

Theorem 2.1 *There does not exist metric structures on the 3-Lie algebra L.*

Proof. Let $B : L \times L \to F$ be the bilinear symmetric form on L which satisfies identity (4). Then by the multiplication of L, we have

$$B(e_2, e_1) = B([e_{m+1}, e_1, e_2], e_1) = B(e_2, [e_{m+1}, e_1, e_1]) = 0,$$

$$B(e_2, e_2) = B([e_{m+1}, e_1, e_2], e_2) = -B(e_1, [e_{m+1}, e_2, e_2]) = 0,$$

$$B(e_2, e_{j-1}) = B(e_2, [e_1, e_2, e_j]) = -B(e_2, [e_1, e_2, e_2]) = 0,\ 4 \le j \le m,$$

$$B(e_2, e_m) = B([e_{m+1}, e_1, e_2], e_m) = -B(e_1, [e_{m+1}, e_2, e_m]) = 0,$$

$$B(e_2, e_{m+1}) = B([e_{m+1}, e_1, e_2], e_{m+1}) = -B(e_1, [e_{m+1}, e_2, e_{m+1}]) = 0.$$

Then, $B(e_2, L) = 0$, that is, B is degenerated. Therefore, there does not exist metric structures on the 3-Lie algebra L.

3 Derivations of the 3-Lie Algebra L

In this section we study the derivations of 3-Lie algebra L. Let $D : L \to L$ be any derivation of L, and suppose the matrix of D in the basis $\{e_1, \cdots, e_{m+1}\}$ is $A = (a_{ij})$, $1 \le i, j \le m+1$, that is, $D = \sum\limits_{j=1}^{m+1} a_{ij} E_{ij}$, where E_{ij} is the matrix unit with the number 1 in the position i^{th}-row and j^{th}-column, $1 \le i, j \le m+1$.

Theorem 3.1 *Let L be the 3-Lie algebra in Lemma 1.1, then the dimension of the inner derivation algebra is $2m-1$, and*

$$\begin{aligned} ad(L) = {} & F\Big(E_{22} + \sum_{k=3}^{m}(m-k+2)E_{kk}\Big) + F\Big(\sum_{k=4}^{m} E_{kk-1} + E_{m+12}\Big) + FE_{m+13} \\ & + F\Big(E_{23} - (m-2)E_{m+14}\Big) + F\Big(E_{2m-1} + \sum_{j=5}^{m-1} E_{jj-2} - 2E_{m+1m}\Big) \\ & + \sum_{j=2}^{m} FE_{1j} + \sum_{j=5}^{m-1} F\Big(E_{2j-1} - E_{mj-2} - (m-j+2)E_{m+1j}\Big). \end{aligned} \tag{5}$$

Proof. For every $1 \le k, l \le m+1$, let $ad(e_k, e_l)(e_i) = \sum\limits_{j=1}^{m+1} a_{ij} e_j$. By the direct computation according to Lemma 1.1, we have

$ad(e_1,e_2)=\sum\limits_{k=4}^{m}E_{kk-1}+E_{m+12}$, $ad(e_{m+1},e_1)=E_{22}+\sum\limits_{k=3}^{m}(m-k+2)E_{kk}$,
$ad(e_{m+1},e_2)=-E_{12}$, $ad(e_1,e_3)=(m-1)E_{m+13}$,
$ad(e_{m+1},e_j)=-(m-j+2)E_{1j}$, $3\le j\le m$; $ad(e_2,e_j)=E_{1j-1}$, $4\le j\le m$;
$ad(e_1,e_4)=-E_{23}+(m-2)E_{m+14}$, $ad(e_j,e_m)=E_{1j-2}$, $5\le j\le m-1$;
$ad(e_j,e_1)=E_{2j-1}-E_{mj-2}-(m-j+2)E_{m+1j}$, $5\le j\le m-1$,
$ad(e_1,e_m)=-E_{2m-1}-\sum\limits_{j=5}^{m-1}E_{jj-2}+2E_{m+1m}$.

Therefore, $ad(e_{m+1},e_j),1\le j\le m$; $ad(e_1,e_j),2\le j\le m$ is a basis of $\mathrm{ad}(L)$, and the dimension of $\mathrm{ad}(L)$ is $2m-1$.

Theorem 3.2 *The inner derivation algebra $ad(L)$ is a non-nilpotent but solvable Lie algebra. And $ad(L)$ can be decomposed into the semi-direct sum $ad(L)=B\dot{+}J$, where $B=\sum\limits_{j=2}^{m+1}ad(e_1,e_j)$ is the solvable subalgebra of $ad(L)$, $J=\sum\limits_{j=2}^{m}ad(e_{m+1},e_j)$ is an abelian ideal of $ad(L)$, and $[B,J]=J$.*

Proof. Since $[ad(e_{m+1},e_k),ad(e_{m+1},e_l)]=0$ for $2\le k,l\le m$, $[J,J]=0$.
By Lemma 1.1 and Theorem 3.1, the products of the basis vectors

$[ad(e_1,e_2),ad(e_{m+1},e_l)]=\frac{-1}{m-l+3}ad(e_{m+1},e_{l+1})+ad(e_{m+1},e_{l-1})$, $4\le l\le m$;

$[ad(e_1,e_k),ad(e_{m+1},e_2)]=0$, $3\le k\le m-1$;

$[ad(e_1,e_m),ad(e_{m+1},e_l)]=2ad(e_{m+1},e_l)+\frac{-1}{m-l+3}ad(e_{m+1},e_{l-1})$, $5\le l\le m-1$;

$[ad(e_1,e_m),ad(e_{m+1},e_2)]=\frac{-2}{3}ad(e_{m+1},e_{m-1})$,

$[ad(e_1,e_k),ad(e_{m+1},e_m)]=\frac{m-k+2}{m-k+4}ad(e_{m+1},e_{k-2})+ad(e_{m+1},e_{k-1})$, $5\le k\le m-1$;

$[ad(e_1,e_k),ad(e_1,e_{m+1})]=(m-k+2)ad(e_1,e_k)$, $3\le k\le m$;

$[ad(e_1,e_2),ad(e_1,e_{m+1})]=ad(e_1,e_2)$,

$[ad(e_1,e_k),ad(e_1,e_2)]=ad(e_1,e_{k-1})$, $4\le k\le m$,

and others are zero.

Then $[B,B]=\sum\limits_{j=2}^{m}Fad(e_1,e_j)\subset B$, $[B,J]=J$, $[J,J]=0$, $B^{(m-1)}=0$, and $(adL)^s=L^1=\sum\limits_{j=2}^{m}Fad(e_1,e_j)+\sum\limits_{j=2}^{m}Fad(e_{m+1},e_j)\neq 0$. Therefore $\mathrm{ad}(L)$ is solvable, but non-nilpotent. It follows the result.

Theorem 3.3 *Let L be the 3-Lie algebra in Lemma 1.1. Then* $\dim DerL=2m+1$, *and*

$DerL=F\Big(E_{m+1m}-\frac{1}{2}E_{2m-1}-\frac{1}{2}\sum\limits_{k=5}^{m-1}E_{kk-2}\Big)+F\Big(E_{22}+\sum\limits_{k=3}^{m}(m-k+2)E_{kk}\Big)$

$$+\sum_{k=2}^{m} FE_{1k} + \sum_{k=4}^{m-2} F\Big(E_{2k} - E_{mk-1} - (m-k+1)E_{m+1k+1}\Big) + FE_{m+13}$$
$$+F\Big(E_{23} - (m-2)E_{m+14}\Big) + F\Big(\sum_{k=4}^{m} E_{kk-1} + E_{m+12}\Big)$$
$$+F\Big(E_{11} + \sum_{k=3}^{m}(m-k+1)E_{kk} - E_{m+1m+1}\Big) + FE_{1m+1}$$
$$= adL + F\Big(E_{11} + \sum_{k=3}^{m}(m-k+1)E_{kk} - E_{m+1m+1}\Big) + FE_{1m+1}. \qquad (6)$$

Proof. Let D be any derivation of the 3-Lie algebra L. Suppose $D(e_i) = \sum_{k=1}^{m+1} a_{ik}e_k, 1 \le i \le m+1$. By Lemma 1.1, for $5 \le j \le m-1$

$D([e_1, e_2, e_j]) = [\sum_{k=1}^{m+1} a_{1k}e_k, e_2, e_j] + [e_1, \sum_{k=1}^{m+1} a_{2k}e_k, e_j] + [e_1, e_2, \sum_{k=1}^{m+1} a_{jk}e_k]$
$= (a_{11} + a_{22})e_{j-1} + a_{2m+1}(m-j+2)e_j + \sum_{k=1}^{m+1} a_{jk+1}e_k + a_{jm+1}e_2 + a_{2m}e_{j-2}$
$= D(e_{j-1}) = \sum_{k=1}^{m+1} a_{j-1k}e_k,$

$D([e_1, e_j, e_m]) = a_{11}e_{j-2} + a_{j2}e_{m-1} + \sum_{k=5}^{m-1} a_{jk}e_{k-2} + 2a_{jm+1}e_m + a_{mm}e_{j-2}$
$+(m-j+2)a_{mm+1}e_j - a_{m2}e_{j-1}= D(e_{j-2}) = \sum_{k=1}^{m+1} a_{j-2k}e_k,$

$D([e_1, e_2, e_4]) = (a_{11} + a_{22})e_3 + \sum_{k=4}^{m} a_{4k}e_{k-1} + a_{4m+1}e_2 = D(e_3) = \sum_{k=1}^{m+1} a_{3k}e_k,$

$D([e_1, e_2, e_m]) = (a_{11} + a_{22})e_{m-1} + \sum_{k=5}^{m-1} a_{2k}e_{k-2} - 2a_{2m+1}e_m + \sum_{k=4}^{m} a_{mk}e_{k-1}$
$+a_{mm+1}e_2 = D(e_{m-1}) = \sum_{k=1}^{m+1} a_{m-1k}e_k,$

$D([e_{m+1}, e_1, e_2]) = (a_{m+1m+1} + a_{11} + a_{22})e_2 + \sum_{k=3}^{m}(m-k+2)a_{2k}e_k$
$+ \sum_{k=4}^{m} a_{m+1k}e_{k-1} = D(e_2) = \sum_{k=1}^{m+1} a_{2k}e_k,$

$D([e_{m+1}, e_1, e_3]) = (m-1)(a_{m+1m+1}+a_{11}+a_{33})e_3+a_{32}e_2+\sum_{k=4}^{m}(m-k+2)a_{3k}e_k$
$= (m-1)D(e_3) = (m-1)\sum_{k=1}^{m+1} a_{3k}e_k,$

$D([e_{m+1}, e_1, e_4]) = [(m-1)a_{43} - a_{m+12}]e_3 + (m-2)(a_{m+1m+1} + a_{11} + a_{44})e_4$
$a_{42}e_2 + \sum_{k=5}^{m}(m-k+2)a_{4k}e_k = (m-2)\sum_{k=1}^{m+1} a_{4k}e_k.$

For every k satisfies $5 \le k \le m-1$, we have

$D([e_{m+1}, e_1, e_k]) = -a_{m+12}e_{k-1} + a_{m+1m}e_{k-2} + (m-k+2)(a_{m+1m+1} + a_{11})e_k$
$+ \sum_{n=3}^{m}(m-n+2)a_{kn}e_n + a_{k2}e_2 = \sum_{n=1}^{m+1}(m-k+2)a_{kn}e_n.$

$D([e_{m+1}, e_1, e_m]) = a_{m2}e_2 - \sum_{k=5}^{m-1} a_{m+1k}e_{k-2} - a_{m+12}e_{m-1} + (2a_{11} + 2a_{m+1m+1})e_m$

$+\sum\limits_{k=3}^{m}(m-k+2)a_{mk}e_k = D(2e_m) = 2\sum\limits_{k=1}^{m+1}a_{mk}e_k,$

$D([e_{m+1}, e_2, e_m]) = a_{m+11}e_{m-1} + 2a_{21}e_m - a_{m2}e_2 = 0.$

Summarizing, we obtain

$a_{kk} = a_{44} - (k-4)(a_{11}+a_{22}), a_{2m} + a_{kk-1} = a_{k-1k-2},\ 5 \le k \le m;$

$a_{kk-1} = a_{m+12}, a_{k-1k} = -(m-k+2)a_{2m+1} + a_{kk+1},\ 2a_{kk-2} = -a_{m+1m},$ $5 \le k \le m-1;$

$a_{k2} = a_{k-2m-1}, a_{kk+1} - a_{m2} = a_{k-2k-1},\ 5 \le k \le m-2;$

$a_{kk+2} + (m-k+2)a_{mm+1} = a_{k-2k},\ 5 \le k \le m-3;$

$a_{m+1k} = -(m-k+2)a_{2k-1}, a_{mk-2} = -a_{2k-1}, 4 \le k \le m-1;$

$a_{jk+1} = a_{j-1k},\ k \ne 2, j, j-1, j-2,\ 3 \le k \le m-1;$

$a_{2k+1} + a_{mk+1} = a_{m-1k},\ 3 \le k \le m-3;\ a_{mm-1} = a_{m+12},\ a_{m+12} = a_{43},$

$a_{m+1m+1} = -a_{11},\ a_{m+11} = a_{33} = a_{11} + a_{22} + a_{44},\ a_{mm} = a_{11} + 2a_{22},$

$a_{44} = (m-3)a_{11} + (m-2)a_{22},\ a_{km+1} = a_{k-2m} = 0,\ 5 \le k \le m-1;$

$a_{kj} = 0,\ 4 \le k \le m-1, j = 1, 2, m, m+1;\ a_{4k} = 0, 5 \le k \le m;$

$a_{3k} = 0, 1 \le k \le m, k \ne 3;$

$a_{m1} = a_{m2} = a_{mm+1} = a_{mm-2} = a_{21} = a_{2m} = a_{2m+1} = 0.$

Therefore, the matrix form of D in the basis $\{e_1, \cdots, e_{m+1}\}$ is

$$\begin{pmatrix}
a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & \cdot & a_{m-3}^1 & a_{m-2}^1 & a_{m-1}^1 & a_m^1 & a_{m+1}^1 \\
0 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdot & a_{m-3}^2 & a_{m-2}^2 & \frac{-1}{2}a_m^{m+1} & 0 & 0 \\
0 & 0 & \delta_3 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_3^4 & \delta_4 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{2}a_m^{m+1} & a_3^4 & \delta_5 & \cdot & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdot & \frac{-1}{2}a_m^{m+1} & a_3^4 & \delta_{m-1} & 0 & 0 \\
0 & 0 & -a_4^2 & -a_5^2 & -a_6^2 & \cdot & -a_{m-2}^2 & 0 & a_3^4 & \delta_m & 0 \\
0 & a_3^4 & a_3^{m+1} & 0 & -a_4^2 & \cdot & -5a_{m-4}^2 & -4a_{m-3}^2 & -3a_{m-2}^2 & a_m^{m+1} & -a_1^1
\end{pmatrix},$$

where $\delta_k = (m-k+1)a_{11} + (m-k+2)a_{22}$, $3 \le k \le m$; $n_k = -(m-k+2)$, $5 \le k \le m-1$. Then for every derivation D,

$D = a_{m+1m}\Big(E_{m+1m} - \frac{1}{2}E_{2m-1} - \frac{1}{2}\sum\limits_{k=5}^{m-1}E_{kk-2}\Big) + a_{22}\Big(E_{22} + \sum\limits_{k=3}^{m}(m-k+2)E_{kk}\Big)$

$+\sum\limits_{k=2}^{m}a_{1k}E_{1k} + \sum\limits_{k=4}^{m-2}a_{2k}\Big(E_{2k} - E_{mk-1} - (m-k+1)E_{m+1k+1}\Big) + a_{m+13}E_{m+13}$

$+a_{23}\Big(E_{23} - (m-2)E_{m+14}\Big) + a_{m+12}\Big(\sum\limits_{k=4}^{m}E_{kk-1} + E_{m+12}\Big)$

$+a_{11}\Big(E_{11} + \sum\limits_{k=3}^{m}(m-k+1)E_{kk} - E_{m+1m+1}\Big) + a_{1m+1}E_{1m+1}.$

The result follows.

Theorem 3.4 *The derivation algebra of L is solvable, and there exist only two exterior derivations*

$$D_1 = E_{11} + \sum_{k=3}^{m}(m-k+1)E_{kk} - E_{m+1m+1}, \quad D_2 = E_{1m+1}.$$

Therefore, $DerL/ad(L) = FE_{1m+1} + F\Big(E_{11} + \sum\limits_{k=3}^{m}(m-k+1)E_{kk} - E_{m+1m+1}\Big)$.

Proof. By Theorem 3.2 and Theorem 3.3 we have

$$DerL^1 = [DerL, DerL] = adL^1 + FE_{1m+1}, DerL^{(2)} = [DerL^1, DerL^1] = adL^1.$$

There exists s, such that $DerL^{(s+1)} = [DerL^1, DerL^1] = adL^{(s)} = 0$. Therefore, $DerL$ is solvable. Since $\mathrm{ad}(L)$ is non-nilpotent, $DerL$ is non-nilpotent.

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References

[1] V. FILIPPOV, n-Lie algebras, *Sib. Mat. Zh.,* 1985, 26 (6), 126-140.

[2] L. TAKHTAJAN, On foundation of the generalized Nambu mechanics [J],*Comm. Math. Phys.,* 160 (1994) 295-315.

[3] P. HO, R. HOU R, and Y. MATSUO, Lie 3-algebra and multiple M_2-branes [J], arXiv: 0804.2110.

[4] L. TKHTAJAN, On foundation of the generalized Nambu mechanics, *Commun. Math. Phys.* 160(1994) 295 [hep-th/9301111].

[5] J. BAGGER and N. LAMBERT, Modeling multiple M2s, Phys. Rev. D75(2007) 045020 [hep-th/0611108]; Gauge symmetry and supersymmetry of multipleM2-branes, *Phys. Rev.* D77 (2008) 065008 [arXiv:0711.0955]; Comments on multiple M2-branes, *JHEP* 02(2008) 105 [arXiv:0712.3738].

[6] R.BAI, C. SHEN and Y. ZHANG, Solvable 3-Lie algebras with a Maximal Hypo-nilpotent ideal $N*$, *Electronic Journal of Linear Algebra,* 2010,21: 43-62.

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