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Strong sequences and independent sets

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Abstract

A family $\mathcal{S} \in \mathcal{P}(\omega)$ is *an independent family* if for each pair $\mathcal{A}, \mathcal{B}$ of disjoint finite subsets of $\mathcal{S}$ the set $\bigcap \mathcal{A} \cap (\omega \setminus \bigcup \mathcal{B})$ is nonempty. The fact that there is an independent family on ω of size continuum was proved by Fichtenholz and Kantorowicz in [7]. If we substitute $\mathcal{P}(\omega)$ by a set (X, r) with arbitrary relation r it is natural question about existence and length of an independent set on (X, r). In this paper special assumptions of such existence will be considered.

On the other hand in 60s' of the last century the strong sequences method was introduced by Efimov. He used it for proving some famous theorems in dyadic spaces like: Marczewski theorem on cellularity, Shanin theorem on a calibre, Esenin-Volpin theorem and others.

In this paper there will be considered: length of strong sequences, the length of independent sets and other well known cardinal invariants and there will be examined inequalities among them.

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1 Introduction

The problem of existence and length of an independent families were considered by several researchers. Fichtenholz and Kantorowicz proved in their paper [7] that there exists an independent family on ω of size continuum. Moreover Hausdorff in [9] obtained such result for any cardinal number. One of other fundamental papers in this topic was published by Balcar and Franek ([1]) where authors presented a proof (without any set-theoretical assumptions) that in each infinite complete Boolean algebra $\mathcal{B}$ there is an independent family $\mathcal{S} \subset \mathcal{B}$ such that $|\mathcal{S}| = |\mathcal{B}|$.

In [3] authors considered different kinds of independent families in order to find the question posted by Eckerton (in [4]): "Can there be, for $\kappa > \omega$, a family $\mathcal{A}$ on κ which is simultaneuosly maximal ω-independent and (maximal) κ-independent?"

In paper [8] there was introduced a cardinal invariant

$$i = \min\{|\mathcal{A}| \colon \mathcal{A} \text{ is a maximal independent family on } X\}.$$

In paper [11] the authors investigated some natural conditions on partial ordered sets and in a family of sets and they proved (under some assumptions) the existence of an independent family in a family of sets $\mathcal{S}$ of cardinality greater than $c(\mathcal{S})$, where $c(\mathcal{S})$ means cellularity of $\mathcal{S}$.

In [10] there was continued investigation of so called strong sequences method. This method was introduced by Efimov in [5] and explored by Turzański (see [17], [18]). (For further historical notes one can go to [10]). The method of strong sequences was crucial for proving some theorems in dyadic spaces like: Marczewski theorem on cellularity, Shanin theorem on a calibre, Esenin-Volpin theorem and others (compare [6], [12], [13], [16]).

In [10] the strong sequences were introduced on a set (X, r) with arbitrary relation r and was defined a new cardinal:

$$\hat{s}(X) = \sup\{\kappa \colon \text{ there exists a strong sequence on } X \text{ of length } \kappa\ \}$$

There were also proposed some inequalities among $\hat{s}(X)$ and other well known invariants like: saturation, calibre, boundeness, density.

The aim of this paper is to examine the existence of various independent sets on a set (X, r) with arbitrary relation r. One can find this paper as continuation of [10], but all needed results to follow the considerations will be quoted here in adequate places.

2 Notation, main definitions and some previous results

All considerations in this paper will be provided on a set (X, r) with arbitrary relation r.

We will sometimes write X instead of (X, r) especially in such situations when it will be obvious which relation we mention. For given X we denote its cardinality by $|X|$. If κ is a cardinal then $[X]^\kappa = \{A \subset X \colon |A| = \kappa\}$. The smallest cardinal number which is greater than κ is its successor and it is denoted by κ^+. Infinite ordinals are usually denoted by Greek letters. The remaining notation is standard.

Below there will be presented some basic definitions, (compare [10]).

We say that a and b are *comparable* if $(a,b) \in r$ or $(b,a) \in r$.

We say that a is *compatible* with b (or a and b are compatible) if there exists c such that

$$(c,a) \in r \text{ and } (c,b) \in r.$$

We say that $\mathcal{L} \subset X$ is a *chain* if any $a, b \in \mathcal{L}$ are comparable.

Let κ be a cardinal number. A set $D \subset X$ is κ- *directed* if every subset of D of cardinality less than κ has a bound, i.e. for each $B \subset D$ with $|B| < \kappa$ there exists $a \in D$ such that $(a,b) \in r$ for all $b \in B$.

We say that a set $\mathcal{A} \subset X$ is an *antichain* if any two distinct elements $a, b \in \mathcal{A}$ are incomparable.

We say that a set $\mathcal{A} \subset X$ is a *strong antichain* if any two distinct elements $a, b \in \mathcal{A}$ are incompatible.

For simplifying notation we will use:
$a \parallel b$ if and only if a, b are compatible
and
$a \perp b$ if and only if a, b are incompatible.

The minimal cardinal κ such that every antichain on X has size less than κ is called *saturation of* X.

The minimal cardinal κ such that every strong antichain on X has size less than κ is called *strong saturation of* X.

For these two invariants we will use the notation $sat(X)$ and $ssat(X)$, respectively.

Considering the definitions above the following lemma is obvious

Lemma 2.1 *For each set (X,r) with relation we have*

$$sat(X) \geq ssat(X).$$

Let us remind the following examples (compare [10]):

Example 2.2 *Let us note that Sierpinski poset (see in [10], [14], [15]) $P = (R, r)$, where $(x,y) \in r \Leftrightarrow x \leq y \wedge x \leq_w y$ ($\leq_w$ means well ordering on real numbers set R) is an example of partially ordered set which has no uncountable chains and nor uncountable antichains, but it is ω- directed. Then $sat(X) \geq ssat(X)$.*

Example 2.3 *Let us consider the set X of all countable sequences consisting of 0 and 1. Such set forms a tree. Then $sat(X) > ssat(X)$.*

In papers [10] and [17] the main role play strong sequences method as one of the combinatorial methods. Below one can find the version of definition of strong sequences proposed in [10].

Definition 2.4 *Let (X, r) be a set with relation r.*
A sequence $(S_\phi, H_\phi); \phi < \alpha$ where $S_\phi, H_\phi \subset X$ and S_ϕ is finite is called a strong sequence if for each $\phi < \alpha$
1^o $S_\phi \cup H_\phi$ is ω directed
2^o $S_\beta \cup H_\phi$ is not ω directed for $\beta > \phi$.

In paper [10] one can find historical notes about strong sequences and proofs of the following version of theorem of strong sequences.

Theorem 2.5 *If for a set (X, r) with relation there exists a strong sequence $(S_\alpha, H_\alpha); \alpha < (\kappa^\lambda)^+$ such that $|H_\alpha| \leq \kappa^\lambda$ for each $\alpha < (\kappa^\lambda)^+$, then there exists a strong sequence $(S_\alpha, T_\alpha); \alpha < (\lambda)^+$ such that $|T_\alpha| < \omega$ for each $\alpha < (\lambda)^+$,*

Corollary 2.6 *If for a set (X, r) with relation r there exists a strong sequence $(S_\alpha, H_\alpha); \alpha < (\kappa^\lambda)^+$ such that $|H_\alpha| \leq \kappa$ for each $\alpha < (\kappa^\lambda)^+$, then there exists a set A of cardinality greater than λ which contains only pairwise incomparable elements.*

Let

$$\hat{s}(X) = sup\{\kappa \colon \text{there exists a strong sequence on } X \text{ of the length } \kappa\}.$$

Following [2] p. 23 we have

Definition 2.7 *A cardinal κ is a calibre for X if κ is infinite and every set $A \in [X]^\kappa$ has a κ^+-directed set.*

Definition 2.8 *A cardinal κ is a precalibre for X if κ is infinite and every set $A \in [X]^\kappa$ has an ω-directed subset of cardinality κ.*

It is easy to show that each calibre is a precalibre, but the converse theorem is not true. As an example we can get Sierpiński poset (quoted above).

For definition of a calibre proposed above the following theorem will be proved (compare [10]).

Theorem 2.9 *Let (X, r) be a set with relation r. Then each regular cardinal number $\kappa > \hat{s}(X)$ is a calibre for X.*

Proof Let us suppose that κ is not a calibre for X. It means that there exists $A \in [X]^\kappa$ of cardinality κ in which each κ^+-directed set has cardinality less than κ.

Let $a_0 \in A$ be an arbitrary element and $A_0 \subset A$ be a maximal κ^+-directed set such that $a_0 \in A_0$. Obviously $|A_0| < \kappa$. Let $(\{a_0\}, A_0)$ be the first pair of a strong sequence.

Let us suppose that the strong sequence $\{(\{a_\beta\}, A_\beta): \beta < \alpha\}$, where $a_\beta \in A \setminus \bigcup_{\gamma<\beta} A_\gamma$ and $A_\beta \subset A \setminus \bigcup_{\gamma<\beta} A_\gamma$ is a maximal κ^+-directed set such that $a_\beta \in A_\beta$, $\beta < \alpha$ has been defined.

Since $A \setminus \bigcup_{\beta<\alpha} A_\beta$ is not empty because $|A_\beta| < \kappa$ and $|\bigcup_{\beta<\alpha} A_\beta| < \kappa$ for all $\beta < \alpha$, hence we can choose an arbitrary element $a_\alpha \in A \setminus \bigcup_{\beta<\alpha} A_\beta$ and a maximal κ^+-directed set $A_\alpha \subset A \setminus \bigcup_{\beta<\alpha} A_\beta$ such that $a_\alpha \in A_\alpha$. Let (a_α, A_α) be the next pair of the strong sequence.

According to construction shown above we have obtained the strong sequence

$$\{(\{a_\alpha\}, A_\alpha): \alpha < \kappa\}$$

of length greater than $\hat{s}(X)$. Contradiction.

Each calibre is a precalibre, hence the following corollary is true

Corollary 2.10 *Let (X, r) be a set with relation r. Then each regular cardinal number $\kappa > \hat{s}(X)$ is a precalibre for X.*

In [10] such theorem was proved too.

Theorem 2.11 *Let (X, r) be a set with relation r. If $sat(X)$ is regular, then $sat(X) \leq \hat{s}(X)$.*

According to lemma 2.1 and theorem 2.11 we can easily obtain

Corollary 2.12 *Let (X, r) be a set with relation. If $ssat(X)$ is regular, then $ssat(X) \leq \hat{s}(X)$.*

3 The main results

The main result of this paper is to find length of a maximal independent set on a considered set (X, r). The definition proposed below is related to adequate given in [11].

Definition 3.1 *A sequence of ordered pairs $\{(x^0_\alpha, x^1_\alpha): \alpha < \kappa\}$ where $x^0_\alpha \perp x^1_\alpha$ is said to be an independent set if for each finite set $F \subset \kappa$ and for each function $i: F \to \{0,1\}$ the set $\{x^{i(\alpha)}_\alpha: \alpha \in F\}$ is $\omega-$ directed.*

In paper [3] authors carried out research with so called $\kappa-$ independent families. The definition of $\kappa-$ independent sets presented below is slightly different.

Definition 3.2 *A sequence of ordered pairs $\{(x^0_\alpha, x^1_\alpha): \alpha < \kappa\}$ where $x^0_\alpha \perp x^1_\alpha$ is said to be a κ-independent set if for each set $F \subset \kappa$ of cardinality less than κ and for each function $i: F \to \{0,1\}$ the set $\{x^{i(\alpha)}_\alpha: \alpha \in F\}$ is $\kappa-$ directed.*

In order to further investigations let us introduce two new cardinal invariants

$$i(X) = \sup\{|A| \colon A \text{ is an independent set in } X\}.$$

$$i_\kappa(X) = \sup\{|A| \colon A \text{ is a } \kappa\text{-independent set in } X\}.$$

In further part of the paper we will obtain some results concerning existence and length of independent sets. The new situation when we provide our results on a set (X, r) with arbitrary relation r make us seek another assumption under which an independent set could exist. It is obvious that we need special construction which will exclude both of trivial situations: linear ordered set, and a set consisted of a strong antichain only. In this sense let us consider the following properties

We say that $S \subset X$ has $A-$ *property* iff for all $x, y \in S$ if $x \perp y$ then there exists $z \in S$ such that $x \parallel z$ and $z \parallel y$.

We say that $S \subset X$ has $Q-$ *property* iff for all $x, y \in S$ if $x \parallel y$ then there exists $z \in S$ such that $x \perp z$ or $z \perp y$.

Example 3.3 *Let us notice that we can easily find an example which fulfills both of above properties. Let us consider a tree T with $|T| \geq \kappa$ with no chains and antichains of cardinality κ. Moreover, an easy example of a set which fulfills $Q-$property but must not fulfills $A-$property is a set of cardinality κ where κ is a precalibre. This easy observation is left to the reader.*

In order to further applications we need more general property. Let us consider the following definitions

We say that (X, r) has $A(\kappa)$ - *property* iff for all $S \subset X$ of cardinality at least κ the following statement is true: for all $x, y \in S$ if $x \perp y$ then

$$|\{z \in S \colon x \parallel z \wedge z \parallel y\}| = \kappa.$$

The following lemma is obvious

Lemma 3.4 *Let κ be a cardinal number. Let (X, r) be a set of cardinality κ with relation r. If X has $A(\kappa)-$property, then $ssat(X) \leq \kappa$.*

Let us prove the following theorem

Theorem 3.5 *Let κ be a regular cardinal number. Let (X, r) be a set with relation r which has $A(\kappa)$- and Q-property. If $|X| = \kappa$ then there exists a $\kappa-$independent set in X of cardinality κ.*

Proof Let $x_0, y_0 \in X$ be arbitrary elements. If $x_0 \perp y_0$, then we take (x_0, y_0) as the first pair of an independent set. If $x_0 \parallel y_0$, then according to Q-property we can find $z_0 \in X$ such that $x_0 \perp z_0$ or $z_0 \perp y_0$. Hence we choose the adequate pair of incompatible elements, name it (x_0, y_0) and take it as the first pair of a κ-independent set. Let us consider a set

$$A_0 = \{z \in X: x_0 \parallel z \wedge z \parallel y_0\}.$$

Obviously $x_0, y_0 \notin A_0$. According to $A(\kappa)$-property $|A_0| = \kappa$.

Let $x_1, y_1 \in A_0$ be a pair of incompatible elements (such elements exists because Q-property). Let

$$A_1 = \{z \in A_0: x_1 \parallel z \wedge z \parallel y_1\}.$$

Obviously $x_1, y_1 \notin A_1$. According to $A(\kappa)-$property $|A_1| = \kappa$.

Let us suppose that the κ-independent set $(x_\gamma, y_\gamma), \gamma < \beta$ and set

$$A_\gamma = \{z \in A_\delta: x_\gamma \parallel z \wedge z \parallel y_\gamma\}, \delta < \gamma < \beta$$

have been defined.

Let us notice that $A_\xi \subset A_\psi$ for $\psi < \xi$ and $|A_\xi| = \kappa$.

Let us consider a set of all subsets of κ of cardinality less than κ, i.e.

$$\mathcal{F} = \{F_\delta \subset \kappa: |F_\delta| < \kappa \text{ for } \delta < \kappa\}.$$

Let

$$F_\beta = \bigcup\{F_\delta: F_\delta \in \mathcal{F} \text{ and } \beta \in F_\delta\}$$

and let

$$\Gamma_\beta = \{\gamma < \beta: \gamma \in F_\beta\}.$$

Let us consider $\bigcap_{\gamma \in \Gamma_\beta} A_\gamma$. Such a set is non-empty because of previous construction and fact that $|A_\gamma| = \kappa$ for $\gamma < \beta$. Let $x_\beta \in \bigcap_{\gamma \in \Gamma_\beta} A_\gamma$ be the arbitrary element. According to Q-property there exists $y_\beta \in \bigcap_{\gamma \in \Gamma_\beta} A_\gamma$ such that $x_\beta \perp y_\beta$. Let (x_β, y_β) be the next pair of the κ-independent set. We have constructed a κ-independent set of cardinality κ.

The corollary below follows immediately from lemma 3.4 and theorem 3.5

Corollary 3.6 *Let κ be a regular cardinal number. Let (X, r) be a set of cardinality κ with relation r which has $A(\kappa)-$ and Q-property. Then $i_\kappa(X) \geq ssat(X)$. Moreover if $|X| > ssat(X)$, then $i_\kappa(X) > ssat(X)$.*

Let us notice that the assumption $\kappa > ssat(X)$ in corollary 3.6 cannot be omitted because without it we can easily find relation for which an independent set would not exist. (We mentioned special case of such sets before theorem 3.5).

Corollary 3.7 *Let κ be a regular cardinal number. Let (X, r) be a set with relation r which has $A(\omega)$- and Q-property. If $|X| = \kappa > ssat(X)$ then there exists an independent set in X of cardinality κ. (In other words $i(X) > ssat(X)$).*

Proof Let $x_0, y_0 \in X$ be arbitrary elements. If $x_0 \perp y_0$, then we take (x_0, y_0) as the first pair of an independent set. If $x_0 \parallel y_0$, then according to Q-property we can find $z_0 \in X$ such that $x_0 \perp z_0$ or $z_0 \perp y_0$. Hence we choose the adequate pair of incompatible elements, name it (x_0, y_0) and take it as the first pair of an independent set. Let

$$A_0 = \{z \in X : x_0 \parallel z \wedge z \parallel y_0\}.$$

According to $A(\omega)$-property $|A_0| = \omega$.

Let us suppose that the independent set (x_γ, y_γ) for $\gamma < \beta < \kappa$ and sets

$$A_\gamma = \{z \in X : x_\gamma \parallel z \wedge z \parallel y_\gamma\}$$

have been defined.

Let us consider a set of all subsets of κ of cardinality less than ω, i.e.

$$\mathcal{F} = \{F_\delta \subset \kappa : |F_\delta| < \omega \text{ for } \delta < \kappa\}.$$

Let

$$F_\beta = \bigcup\{F_\delta : F_\delta \in \mathcal{F} \text{ and } \beta \in F_\delta\}$$

and let

$$\Gamma_\beta = \{\gamma < \beta : \gamma \in F_\beta\}.$$

Let us consider $\bigcap_{\gamma \in \Gamma_\beta} A_\gamma$. Such a set is non-empty because of previous construction and fact that $|A_\gamma| = \omega$ for $\gamma < \beta$. Let $x_\beta \in \bigcap_{\gamma \in \Gamma_\beta} A_\gamma$ be the arbitrary element. According to Q-property there exists $y_\beta \in \bigcap_{\gamma \in \Gamma_\beta} A_\gamma$ such that $x_\beta \perp y_\beta$. Let (x_β, y_β) be the next pair of the independent set. We have constructed an independent set of cardinality κ.

One can find below the theorem which connects length of independent set with a precalibre. Although it is not mentioned explicite the method of strong sequences will be used in the proof of the next theorem.

Theorem 3.8 *Let $\kappa > ssat(X)$ be a regular, cardinal number. Let (X, r) be a set with relation r with Q-property such that $|X| \geq (\kappa^{ssat(X)})^+$, where κ is a precalibre of X. Then there exists an independent set of cardinality κ. (In other words $i(X) > \hat{s}(X)$).*

Proof Let us notice that for each element $x \in X$ there exists an element $y \in X$ such that $x \perp y$. It follows from Q-property.

Let us consider the set

$$\mathcal{X} = \{(x, y) \in X \times X : x \perp y\}.$$

Let us suppose that each independent set in $\mathcal{X}$ has cardinality less than κ. Let us choose an arbitrary element from $\mathcal{X}$ and name it (x_0, y_0). Let $A_0 \subset \mathcal{X}$ be a maximal independent set such that $(x_0, y_0) \in A_0$. Hence $|A_0| < \kappa$. Let $((x_0, y_0), A_0)$ be the first element of a strong sequence.

Let us suppose that the strong sequence $((x_\gamma, y_\gamma), A_\gamma)$ for $\gamma < \beta$, where $A_\gamma \subset \mathcal{X} \setminus \bigcup_{\delta<\gamma} A_\delta$ is a maximal independent set such that $(x_\gamma, y_\gamma) \in A_\gamma$ have been defined. Obviously $|A_\gamma| < \kappa$ and $|\bigcup_{\gamma<\beta} A_\gamma| < \kappa$ for each $\gamma < \beta < \kappa$. Hence the set $\mathcal{X} \setminus \bigcup_{\gamma<\beta} A_\gamma$ is nonempty.

Let $(x_\beta, y_\beta) \in \mathcal{X} \setminus \bigcup_{\gamma<\beta} A_\gamma$ be an arbitrary element and $A_\beta \subset \mathcal{X} \setminus \bigcup_{\gamma<\beta} A_\gamma$ be a maximal independent set such that $(x_\beta, y_\beta) \in A_\beta$.

Let $((x_\beta, y_\beta), A_\beta)$ be the next pair of the strong sequence. We have constructed the strong sequence of cardinality κ.

According to corollary 2.6 there exists a set of cardinality $(ssat(X))^+$ consisting of pairwise incompatible elements. Contradiction.

Let us remind that

$$\hat{s}(X) = sup\{\kappa : \text{there exists a strong sequence on } X \text{ of the length } \kappa\}.$$

The following corollaries are easy consequences of previous results.

Corollary 3.9 *Let (X, r) be a set with relation r such that*
1) $|X| > (\kappa^{ssat(X)})^+$, where $\kappa > \hat{s}(X)$
2) X fulfills Q-property.
Then there exists an independent set of cardinality κ.

Proof According to corollary 2.10 we have κ is a precalibre, according to theorem 3.5 we obtain our claim.

Corollary 3.10 *Let (X, r) be a set with relation r such that*
1) $|X| > (\kappa^{ssat(X)})^+$, where $\kappa > \hat{s}(X)$ and $ssat(X)$ be a regular number
2) X fulfills Q-property.
Then

$$ssat(X) \leq \hat{s}(X) < i(X).$$

Proof According to corollary 2.10 κ is a precalibre, according to corollary 2.12 $ssat(X) \leq \hat{s}(X)$. According to corollary 3.9 we obtain the required inequalities.

The next theorem will gives us new inequalities between invariants considered in this paper.

Theorem 3.11 *Let $\hat{s}(X)$ be a regular cardinal number. Let (X, r) be a set with relation r which has $A(\hat{s}(X))$-property. Then $i_{\hat{s}(X)}(X) \geq \hat{s}(X)$.*

Proof Let $(S_\alpha, H_\alpha)_{\alpha<\kappa}$ be a strong sequence of length $\kappa = \hat{s}(X)$. Let $x_0 \in H_0$ be an arbitrary element. Then according to definition 2.4 there is $y_0 \in S_1$ such that $x_0 \perp y_0$. Let (x_0, y_0) be the first pair of a $\hat{s}(X)$- independent set.

Let us consider a set

$$A = \{a \in \bigcup_{\alpha<\kappa} H_\alpha \colon x_0 \parallel a \text{ and } a \parallel y_0\}.$$

Obviously $x_0, y_0 \notin A$. According to $A(\kappa)$-property $|A| = \kappa$. Let us choose an arbitrary element of A and name it x_1. This element belongs to some H_α for $\alpha < \kappa$. Let us consider

$$I_1 = \{\alpha < \kappa \colon x_1 \in H_\alpha\}$$

and let us take $\alpha_1 = \min I_1$. Then according to definition 2.4 for $\alpha > \alpha_1$ there are some elements in each S_α which are incompatible with x_1. Let us denote the set of such elements by M_{α_1}, i.e.

$$M_{\alpha_1} = \{z \in \bigcup_{\alpha>\alpha_1} S_\alpha \colon z \perp x_1\}$$

Such a set has cardinality κ. Let us consider $B_1 = A \cap M_{\alpha_1}$ If B_1 is nonempty then we choose an arbitrary element $y_1 \in B_1$. If $B_1 = \emptyset$ then we can consider

$$B'_1 = \{b \in A \cup M_{\alpha_1} \colon b \text{ is a bound of } S_\alpha \cup H_\alpha, \alpha > \alpha_1\}$$

Let (x_1, y_1) be the second pair of the $\hat{s}(X)$-independent set.

Let us assume that for $\eta < \kappa$ the independent set $(x_\beta, y_\beta)_{\beta<\eta}$ and sets

$$I_\beta = \{\alpha < \kappa \colon x_\beta \in H_\alpha\},$$

where $\alpha_\beta = \min I_\beta$, $B_\beta = A \cap M_{\alpha_\beta}$ where

$$M_{\alpha_\beta} = \{z \in \bigcup_{\alpha>\alpha_\beta} S_\alpha \colon z \perp x_\beta\}$$

and B_β or

$$B'_\beta = \{b \in A \cup M_{\alpha_\beta} \colon b \text{ is a bound of } S_\alpha \cup H_\alpha, \alpha > \alpha_\beta\}$$

have been defined. Obviously

$$A \setminus \{x_\beta \colon \beta < \eta\} \neq \emptyset.$$

Let us choose an arbitrary element $x_\eta \in A \setminus \{x_\beta : \beta < \eta\}$. Let

$$I_\eta = \{\alpha < \kappa : x_\eta \in H_\alpha\}$$

and let $\alpha_\eta = \min I_\eta$. Then according to definition 2.4 in each set S_α, where $\alpha > \alpha_\eta$ there are elements which are incompatible with x_η. Let us denote the set of all such elements by M_{α_η}, i.e.

$$M_{\alpha_\eta} = \{z \in \bigcup_{\alpha > \alpha_\eta} S_\alpha : z \perp x_\eta\}.$$

Let us consider $B_\eta = A \cap M_{\alpha_\eta}$. If B_η is nonempty we choose arbitrary an element from B_η and name it y_η. If $B_\eta = \emptyset$ then we take

$$B'_\eta = \{b \in A \cup M_{\alpha_\eta} : b \text{ is a bound of } S_\alpha \cup H_\alpha, \alpha > \alpha_\eta\}$$

Let (x_η, y_η) be the next pair of the independent set. We have constructed the $\hat{s}(X)$- independent set of cardinality κ.

In paper [3] authors answered the question posted by Eckerton in [4]. (This question was quoted in the introduction of this paper). Although definitions of independent sets in our paper are slightly different the question is if we can obtain the similar result, (compare [3]), i.e.

Question Let $\kappa \geq \omega$ be a regular cardinal number and (X, r) be a set of cardinality at least κ (with relation r). If (X, r) has $A(\kappa)-$ and Q-property then there exists a set $A \subset X$ of cardinality κ which is both a maximal κ-independent set and a maximal independent set. In other words $i_\kappa(X) = i(X)$.

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