Mathematica Aeterna, Vol. 5, 2015, no. 3, 515 - 520

Stable Analysis for a Class of Multi-Step Iteration Scheme

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Abstract

In this paper we consider a multi-step iteration scheme derived from the numerical solutions for systems of linear equations. By making use of the inequality method for vectors, we get a couple of criteria to guarantee the computing stability, including boundedness and convergence.

Mathematics Subject Classification: 65Q10, 65L10.

Keywords: Multi-step; Iteration; Stability; Invariant set; Spectral radius.

1 Introduction

As is well known, there had been celebrated means to solve the numerical solutions for the system of the form

$$x = Bx + f, \tag{1}$$

such as the Jacobi method, Gauss-Seidel method and the successive over-relaxation method, see [3] for the details. We observe that all the methods mentioned above are single step, that is, the new approximation $x(k+1)$ depends only on the previous approximation $x(k)$. For example, the Jacobi iteration is given by

$$x(k+1) = Bx(k) + f,\ k = 0, 1, 2, \ldots .$$

To use the previous values much more, we now consider a multi-step scheme as follows

$$x(k+1) = (1-\omega)x(k) + \omega B(\alpha x(k) - \beta x(k-1)) + \omega f,\ k = 0, 1, 2, \cdots, \tag{2}$$

where $\omega \in (0, 1)$, $\alpha > 0$ and $\beta \geq 0$ with $\alpha = \beta + 1$, $B \in \mathbb{R}^{n\times n}$, $f \in \mathbb{R}^n$ and n is the order of (1).

For any given initial vectors $x(-1) = x_{-1}, x(0) = x_0 \in \mathbb{R}^n$, there exists a unique vector sequence $\{x(k)\}_{k\geq 1}$ defined by (2), which will be called a solution of (2). In general, the solution of (2) may be viewed as $\{x(k)\}_{k\geq -1}$, or $\{x(k)\}$ for short. Note that the solution of (2) depends on the initial vectors. To reflect this relationship, we introduce some symbols as follows. Let the integer set $\{a, a+1, a+2, ..., b\}$ be denoted by $\mathbb{Z}[a, b]$ and $\{a, a+1, a+2, ...\}$ by $\mathbb{Z}[a, \infty)$. Let $\varphi : \mathbb{Z}[-1, 0] \to \mathbb{R}^n$ and $C(\mathbb{Z}[-1, 0], \mathbb{R}^n)$ denote the set of all such φs. Then, when the initial vectors $x_{-1} = \varphi(-1), x_0 = \varphi(0)$, the corresponding solution of (2) can be represented by $\{x(k; \varphi)\}_{k\geq -1}$. For convenience, $\{x(k; \varphi)\}_{k\geq -1}$ is called a solution of (2) through φ.

Throughout this paper, we will make use of the conceptions of absolute values and inequalities in $\mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$). Let $x, y \in \mathbb{R}^n$ and $x = (x_1, x_2, .., x_n)^T, y = (y_1, y_2, .., y_n)^T$. Then the symbol $|x|$ means the vector $|x| = (|x_1|, |x_2|, .., |x_n|)^T$ and $x \leq y$ indicates $x_i \leq y_i$ for all $i \in \mathbb{Z}[1, n]$. The symbols $x < y, x \geq y$, etc. have similar meanings, and $x \leq y$ (or $x < y$) in $\mathbb{R}^{n\times n}$ can be defined likewise.

Let I be an identity matrix. In the sequel the inequality property of matrices is important, we quote as follows.

Lemma 1.1 *[2] Let $A \in \mathbb{R}^{m\times m}$ and $\rho(A)$ denote the spectral radius of A. If $A \geq 0$ and $\rho(A) < 1$, then $I - A$ is inverse and $(I - A)^{-1} \geq 0$.*

Definition 1.2 *[4] Let $\mathbb{S} \subset \mathbb{R}^n$ be bounded. If for any $\varphi \in C(\mathbb{Z}[-1, 0], \mathbb{S})$, the solution $\{x(k; \varphi)\}$ of (2) satisfies that $x(k; \varphi) \in \mathbb{S}$ for all $k \in \mathbb{Z}[-1, \infty)$, then the set $\mathbb{S}$ is said to be invariant of (2).*

Definition 1.3 *[4] Let $\{x(k; \varphi_0)\}$ be a solution of (2) through φ_0. If for any $\varphi \in C(\mathbb{Z}[-1, 0], \mathbb{R}^n)$, the solution $\{x(k; \varphi)\}$ of (2) satisfies that*

$$\varphi \to \varphi_0 \Rightarrow x(k; \varphi) \to x(k; \varphi_0) \textit{ for all } k \in \mathbb{Z}[-1, \infty),$$

then $\{x(k; \varphi_0)\}$ is said to be stable. If for any φ_0, $\{x(k; \varphi_0)\}$ is stable, then (2) is said to be stable.

2 Main Results

Next we discuss the convergence and the computing stabilities for (2). Before doing so, we note that (2) can be rewritten in the difference manner[1] as

$$\Delta x(k) = -\omega x(k) + \omega B(\alpha x(k) - \beta x(k - 1)) + \omega f. \tag{3}$$

Then, for any $\varphi \in C(\mathbb{Z}[-1,0], \mathbb{R}^n)$, the corresponding solution $\{x(k;\varphi)\}_{k\geq -1}$ of (2) satisfies that

$$x(k) = (1-\omega)^k \varphi(0) + \omega \sum_{s=0}^{k-1} (1-\omega)^{k-s-1} \left[\alpha B x(s) - \beta B x(s-1) + f\right]. \quad (4)$$

The formula (4) can be verified straightforwardly. Indeed, we have from (4) that

$$\begin{aligned}
\Delta x(k) &= x(k+1) - x(k) \\
&= (1-\omega)^{k+1}\varphi(0) + \omega \sum_{s=0}^{k} (1-\omega)^{k-s} \left[\alpha B x(s) - \beta B x(s-1) + f\right] \\
&\quad - \left((1-\omega)^k \varphi(0) + \omega \sum_{s=0}^{k-1} (1-\omega)^{k-s-1} \left[\alpha B x(s) - \beta B x(s-1) + f\right] \right) \\
&= -\omega(1-\omega)^k \varphi(0) + \omega B(\alpha x(k) - \beta x(k-1)) + \omega f \\
&\quad -\omega^2 \sum_{s=0}^{k-1} (1-\omega)^{k-s-1} \left[\alpha B x(s) - \beta B x(s-1) + f\right] \\
&= -\omega x(k) + \omega B(\alpha(x(k) - \beta x(k-1)) + \omega f,
\end{aligned}$$

which means our assertion (4) holds.

Theorem 2.1 *Suppose that* $B \in \mathbb{R}^{n\times n}$ *with* $\rho(|B|) < \frac{1}{\alpha+\beta}$. *Then,*
(i) $\mathbb{S} = \{s \in \mathbb{R}^n : |s| \leq [I - (\alpha+\beta)|B|]^{-1}|f|\}$ *is an invariant set of (2);*
(ii) the solution $\{x(k;\varphi)\}$ *of (2) is bounded for any* $\varphi \in C(\mathbb{Z}[-1,0], \mathbb{R}^n)$.

Proof. (i) Let

$$U = [I - (\alpha+\beta)|B|]^{-1}|f|. \quad (5)$$

Since $\rho(|B|) < \frac{1}{\alpha+\beta}$, Lemma 1.1 implies that $U \geq 0$, which means $\mathbb{S} \neq \phi$.

Suppose to the contrary that, there exists an $m \in \mathbb{Z}[1,\infty)$ and a solution $\{x(k;\varphi)\}$ of (2) through $\varphi \in \mathbb{S}$ such that

$$|x(k;\varphi)| \leq U \text{ for all } k \in \mathbb{Z}[-1, m-1] \quad (6)$$

and some component $x_v(m;\varphi)$ of $x(m;\varphi)$ satisfies

$$|x_v(m;\varphi)| > U_v, \quad (7)$$

where U_v denote the v-th component of U. Then, by (4) we have

$$\begin{aligned}
|x(m;\varphi)| &\leq (1-\omega)^m |\varphi(0)| + [1 - (1-\omega)^m][(\alpha+\beta)|B|U + |f|] \\
&= (1-\omega)^m U - (1-\omega)^m \left[(\alpha+\beta)|B|U + |f|\right] + (\alpha+\beta)|B|U + |f| \\
&= [I - (\alpha+\beta)|B|]^{-1}|f|,
\end{aligned}$$

which contradicts (7). Hence, $|x(k;\varphi)| \leq U$ for all $k \in \mathbb{Z}[-1,\infty)$ when $\varphi \in C(\mathbb{Z}[-1,0],\mathbb{S})$. In other words, we have proven that $\mathbb{S}$ is an invariant set of (2).

(ii) Let U be defined as in (5). To prove the second, we first note that we can find a vector $\widetilde{f} \in \mathbb{R}^n$ with $|\widetilde{f}| > 0$ such that $|f| \leq |\widetilde{f}|$. For simplicity, we assume that $|f| > 0$, then $U > 0$. For given $\varphi \in C(\mathbb{Z}[-1,0],\mathbb{R}^n)$, there exists a $\lambda \geq 1$ such that $|\varphi(\theta)| \leq \lambda U$ for $\theta \in \mathbb{Z}[-1,0]$. Next we show that $|x(k;\varphi)| \leq \lambda U$ for all $k \in \mathbb{Z}[-1,\infty)$. Otherwise, there exists $m \in \mathbb{Z}[1,\infty)$ such that

$$|x(k;\varphi)| \leq \lambda U \text{ for all } k \in \mathbb{Z}[-1,m-1] \tag{8}$$

but

$$|x_v(m;\varphi)| > \lambda U_v, \tag{9}$$

where the subscript has the same meaning as above. Then, similar to the proof of (i), we reach that

$$\begin{aligned}
&|x(m;\varphi)| \\
\leq\ & (1-\omega)^m|\varphi(0)| + [1-(1-\omega)^m](\lambda(\alpha+\beta)|B|U + \lambda|f|) \\
\leq\ & (1-\omega)^m\lambda U - (1-\omega)^m(\lambda(\alpha+\beta)|B|U + \lambda|f|) + \lambda(\alpha+\beta)|B|U + \lambda|f| \\
=\ & \lambda U,
\end{aligned}$$

which is contrary to (9). As thus, $\{x(k;\varphi)\}$ is bounded. The proof is complete.

Theorem 2.2 *Suppose that $\rho(|B|) < \frac{1}{\alpha+\beta}$. Then any solution $\{x(k;\varphi)\}$ of (2) converges to the root x^* of (1).*

Proof. In the following discussion, we view the term $x(k;\varphi)$ of $\{x(k;\varphi)\}$ as $x(k)$.

Note that Theorem 2.1(ii) implies that $\{x(k)-x^*\}$ is bounded. Hence we can set

$$\begin{aligned}
& \limsup_{k\to\infty} |x(k)-x^*| \\
=\ & (\limsup_{k\to\infty} |x_1(k)-x_1^*|, \limsup_{k\to\infty} |x_2(k)-x_2^*|, ..., \limsup_{k\to\infty} |x_n(k)-x_n^*|)^T \\
=\ & \overline{x}.
\end{aligned}$$

Now from (1)–(2) we have

$$\begin{aligned}
& x(k+1)-x^* \\
=\ & (1-\omega)(x(k)-x^*) + \omega\alpha B(x(k)-x^*) - \omega\beta B(x(k-1)-x^*),
\end{aligned}$$

which yields that

$$\overline{x} \leq (1-\omega)\overline{x} + \omega|B|(\alpha+\beta)\overline{x},$$

and this results in

$$(I - (\alpha + \beta)|B|)\overline{x} \leq 0. \tag{10}$$

Now invoking $\rho(|B|) < \frac{1}{\alpha+\beta}$, from (10) we learn $\overline{x} = 0$ and therefore, $\lim_{k\to\infty} x(k) = x^*$, which ends our proof.

Theorem 2.3 *Suppose that $\rho(|B|) < \frac{1}{\alpha+\beta}$. Then (2) is stable.*

Proof. For any given $\varphi_0 \in C(Z[-1, 0], \mathbb{R}^n)$, we consider the stability of the solution $\{x(k; \varphi_0)\}$ of (2). For this purpose,we employ the other solution $\{x(k; \varphi)\}$ of (2) through $\varphi \in C(\mathbb{Z}[-1, 0], \mathbb{R}^n)$. In view of (4) it follows that

$$\begin{aligned} & x(k;\varphi) - x(k;\varphi_0) \\ = \; & (1-\omega)^k(\varphi(0) - \varphi_0(0)) \\ + \; & \omega \sum_{s=0}^{k-1} (1-\omega)^{k-s-1} \left\{ \alpha B[x(s;\varphi) - x(s;\varphi_0)] - \beta B[x(s-1;\varphi) - x(s-1;\varphi_0)] \right\}. \end{aligned} \tag{11}$$

For any vector $\varepsilon \in \mathbb{R}^n$ with $\varepsilon > 0$, let

$$U(\varepsilon) = [I - (\alpha + \beta)|B|]^{-1}\varepsilon.$$

Then, our hypothesis $\rho(|B|) < \frac{1}{\alpha+\beta}$ implies $U(\varepsilon) \geq 0$. We assert that $|\varphi(\theta) - \varphi_0(\theta)| \leq U(\varepsilon)$ for $\theta \in \mathbb{Z}[-1, 0]$ induces that $|x(k; \varphi) - x(k; \varphi_0)| \leq U(\varepsilon)$ for all $k \in \mathbb{Z}[-1, \infty)$. Otherwise, there exists $m \in \mathbb{Z}[1, \infty)$ such that

$$|x(k; \varphi) - x(k; \varphi_0)| \leq U(\varepsilon), k \in \mathbf{Z}[0, m-1] \tag{12}$$

and the component $x_v(m; \varphi) - x_v(m; \varphi_0)$ of $x(m; \varphi) - x(m; \varphi_0)$ satisfies that

$$|x_v(m; \varphi) - x_v(m; \varphi_0)| > U_v(\varepsilon), \tag{13}$$

Analogous to the proof in Theorem 2.1, it follows fromt (11) that

$$\begin{aligned} & |x(m;\varphi) - x(m;\varphi_0)| \\ \leq \; & (1-\omega)^m|\varphi(0) - \varphi_0(0)| + [1 - (1-\omega)^m](|B|U(\varepsilon) + \varepsilon) \\ = \; & (1-\omega)^m U(\varepsilon) - (1-\omega)^m[(\alpha+\beta)|B|U(\varepsilon) + \varepsilon] + (\alpha+\beta)|B|U(\varepsilon) + \varepsilon \\ = \; & [I - (\alpha+\beta)|B|]^{-1}\varepsilon, \end{aligned}$$

which contradicts (13). Since the ε is arbitrary, we have proved that

$$\varphi \to \varphi_0 \Rightarrow \; x(k;\varphi) \; \to \; x(k;\varphi_0) \text{ for all } k \in \mathbb{Z}[-1, \infty).$$

Note that φ is arbitrary, the proof is complete.

We remark that if the solution $\{x(k;\varphi)\}$ of (2) converges to the root x^* of (1), then, it follows from Theorem 2.1–2.2 that

$$|x^*| \le [I-(\alpha+\beta)|B|]^{-1}|f|.$$

Next we end up our discussions with an example.

Example 2.4 *Suppose in (1) that*

$$B=\begin{bmatrix} 0 & 1 \\ 0 & 1/3 \end{bmatrix}, \quad f=\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we choose

$$\omega = 0.995, \quad \alpha = 1.2, \quad \beta = 0.2$$

and take the initial values

$$x(-1)=\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad x(0)=\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then

$$[I-(\alpha+\beta)|B|]^{-1}|f| = \begin{pmatrix} 29/8 \\ 15/8 \end{pmatrix}, \quad \rho(|B|)=\frac{1}{3}<\frac{1}{\alpha+\beta}$$

and, by (2) we obtain

$$x(1)=\begin{pmatrix} 2.0945 \\ 1.3648 \end{pmatrix}, \ x(2)=\begin{pmatrix} 2.4361 \\ 1.4787 \end{pmatrix}, \ x(3)=\begin{pmatrix} 2.5011 \\ 1.5004 \end{pmatrix} \to \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix},$$

which verifies our observation.

ACKNOWLEDGEMENTS. Thank the NNSF of China (no. 11271379) for his support in this research.

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Received: June, 2015