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# Some results on the stationary of random fuzzy renewal processes

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**Abstract**

The global of our research is the stationary of random fuzzy renewal process. We will show that if the renewal process be a random fuzzy delayed renewal process then it is stationary process. A theorem about the stationary of random fuzzy renewal process is proved and some results are stated. Also, we show that if a random fuzzy renewal process is not delayed, then it can be stationary if it is a random fuzzy poisson process.

**Keywords**: Random fuzzy renewal processes, Stationary process, Delayed process ,Poisson process.

## 1 Introduction

A lot of research has been done on stochastic renewal process [1-3]. In stochastic renewal process, the interarrival times are random variables. In fuzzy case, the interarrival times of a random fuzzy renewal process are independent and identically distributed (iid) random fuzzy variables. Zhao, Tang, Yun [4] considered a renewal process that the interarrival times were independent and identically distributed fuzzy random variables and prove some theorem like random fuzzy elementary renewal theorem, random fuzzy Blackwell theorem,... The global of our research is the stationary of random fuzzy renewal process. Some people investigated about stationary renewal process, (see [5-6]). Fuzzy

renewal models are frequently used as tools for performance analysis of real-life systems. Although, the characteristics of such systems may vary in time, analytical models typically use random fuzzy stationary renewal processes. However, many situations exist where the assumptions of stationary arrivals, and of Poisson arrivals, will be inaccurate. Now, we consider the conditions that a random fuzzy renewal process can be stationary. We will show that a random fuzzy delayed renewal process is stationary.
However, a random fuzzy delayed renewal process permits the first interarrival time to have a different distribution from the remaining ones. This requires an extended theory to address both the renewal number and the interarrival times. Some important results such as the fuzzy elementary renewal theorem, fuzzy Blackwell renewal theorem and fuzzy Smith key renewal theorem have been established for delayed renewal processes [7].
In section 2, we introduce some basic definitions about fuzzy set theory, Cridibility measure and random fuzzy variables. In section 3, we discuss on random fuzzy delayed renewal process, consider the stationary of random fuzzy renewal process, some theorems and results concerning the stationary of random fuzzy renewal process is proved.

## 2 Definitions and preliminaries

Fuzzy set theory was introduced by Zadeh [8]. He proposed a membership function for modeling fuzzy phenomena. In this section, we state some definitions about fuzzy theory.

**Definition 2.1** *Let $X$ is a universal set, a fuzzy subset $\tilde{A}$ of $X$ is defined by its membership function $\xi_{\tilde{A}} : X \to [0,1]$. The $\alpha$-level set of $\tilde{A}$ is denoted by $\tilde{A}_\alpha = \{x : \xi_{\tilde{A}}(x) \geq \alpha\}$, where $\tilde{A}_0$ is the closure of the set $\{x : \xi_{\tilde{A}}(x) \neq 0\}$.*

Let $\Theta$ be a nonempty set, and $P$ the power set of $\Theta$ (i.e., the largest $\sigma$-algebra over $\Theta$). Each element in $P$ is called an *event*. In order to present an axiomatic definition of credibility, it is necessary to assign to each event $A$ a number $Cr\{A\}$ which indicates the credibility that $A$ will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms [9]:
**Axiom 1.** (*Normality*) $Cr\{\Theta\} = 1$.
**Axiom 2.** (*Monotonicity*)$Cr\{A\} \leq Cr\{B\}$ for $A \subset B$.
**Axiom 3.** (*Self-Duality*) $Cr\{A\} + Cr\{A^c\} = 1$ for any event $A$.
**Axiom 4.** (*Maximality*) $C\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

**Definition 2.2** *The set function $Cr$ is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms. Then* *the triplet* $(\Theta, P, Cr)$ *is called a credibility space.*

*Product credibility measure may be defined in multiple ways. We accept the following axiom.*

**Axiom 5.** (*Product Credibility Axiom*) Let $\Theta_k$ be nonempty sets on which $Cr_k$ are credibility measures, $k = 1, 2, ..., n$, respectively, and $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$. Then

$$Cr\{(\theta_1, \theta_2, ..., \theta_n)\} = Cr_1\{\theta_1\} \wedge Cr_2\{\theta_2\} \wedge ... \wedge Cr_n\{\theta_n\} \tag{1}$$

for each $(\theta_1, \theta_2, ..., \theta_n) \in \Theta$.
Let $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$ be credibility spaces, $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$ and $Cr_1 \wedge Cr_2 \wedge ... \wedge Cr_n$. Then $(\Theta, P, Cr)$ is called the product credibility space of $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$.

**Definition 2.3** *Let* $\xi$ *be a fuzzy variable defined on the credibility space* $(\Theta, P(\Theta), Cr)$. *Then the set*

$$\xi_\alpha(r) = \{\xi(\theta) | \theta \in \Theta, Cr\{\theta\} \geq \alpha\}, \tag{2}$$

*is called* $\alpha$*-level set of* $\xi$.

**Definition 2.4** *Let* $\xi$ *be a fuzzy variable defined on the credibility space* $(\Theta, P(\Theta), Cr)$, *and* $\alpha \in (0, 1]$. *Then*

$$\xi'_\alpha = \inf\{r | Cr\{\xi \leq r\} \geq \alpha\}, \xi''_\alpha = \sup\{r | Cr\{\xi \geq r\} \geq \alpha\}, \tag{3}$$

*are called* $\alpha$*-pessimistic value and* $\alpha$*-optimistic value of* $\xi$*, respectively.*

**Definition 2.5** *(Liu and Liu [10]). Let* $\xi$ *be a fuzzy variable on the credibility space* $(\Theta, P(\Theta), Cr)$. *Then, the expected value* $E[\xi]$ *is defined by:*

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr, \tag{4}$$

*provided that at least one of the two integrals is finite.*
*In particular, if the fuzzy variable* $\xi$ *is positive (i.e.* $Cr\{\xi \leq 0\} = 0$*), then*

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr. \tag{5}$$

**Proposition 2.1** Let $\xi$ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Cr)$. Then we have

$$E[\xi] = \frac{1}{2}\int_0^1 [\xi'_\alpha + \xi''_\alpha]d\alpha. \tag{6}$$

***Proof*** If $\xi$ is normalized, i.e., there exists a real number $r_0$ such that $\mu_\xi(_0) = 1$ and if $r_0 > 0$, then the equation (4) can be rewritten as

$$E[\xi] = \frac{1}{2}[r_0 + \int_{r_0}^{+\infty} Cr(\xi \geq r)dr + r_0 - \int_{-\infty}^{r_0} Cr(\xi \leq r)dr] \tag{7}$$

$$= \frac{1}{2}\int_0^1 (\xi'_\alpha + \xi''_\alpha)d\alpha, \tag{8}$$

The same result can be obtained when $r_0 \leq 0\bullet$

**Definition 2.6** *A random fuzzy variable is a function from the credibility space* $(\Theta, P(\Theta), Cr)$ *to the set of random variables.*

# 3 Stationary random fuzzy renewal process

Consider a renewal process $\{N(t), t \geq 0\}$, that the interarrival times, $\xi$, are independent and same distribution random fuzzy variables. If the first random fuzzy renewal time $\xi_1$ does not have the same distribution as the other $\xi_n$, for $n \geq 2$, We call $N(t)$ a *random fuzzy delayed renewal process*. In other words [11], let $\xi_1, ..., \xi_n$ be known as the fuzzy interarrival times, be nonnegative mutually independent random fuzzy variables defined on the credibility spaces $(\Theta, P(\Theta), Cr)$. Suppose that $\xi_2, ..., \xi_n$ have a common distribution, and that

$$N(t) = \max_{n \geq 0}\{n | 0 \leq \xi_1 + ... + \xi_n \leq t\}. \tag{9}$$

The process $\{N(t), t \geq 0\}$ is called a *random fuzzy delayed renewal process* defined on the credibility space $(\Theta, P(\Theta), Cr)$, and $N(t)$ is a renewal number.

Now, we return to definition of stationary of random fuzzy processes. A fuzzy continuous time stochastic process $\{\tilde{X}(t) : t \geq 0\}$ on credibility space has stationary increment if for any given $t > 0$, the $\tilde{X}(h+t) - \tilde{X}(h)$ are identically distributed fuzzy variables for all $h > 0$. In other words, a fuzzy continuous-time stochastic process is stationary if its finite-dimensional distribution are invariant under any shift in time i.e. for each $0 \leq h_1 < ... < h_k$ and $t \geq 0$,

$$(\tilde{X}(h_1 + t), ..., \tilde{X}(h_k + t)) \equiv (\tilde{X}(h_1), ..., \tilde{X}(h_k)). \tag{10}$$

**Remark 3.1** A fuzzy Markov process $\tilde{X}(t)$ is stationary if $\tilde{X}(t) \equiv \tilde{X}(0), t \geq 0$.

We consider a random fuzzy point process $\tilde{N}(t) = \sum_{n=1}^{\infty} 1(T_n \leq t)$ with points at $0 < T_1 < T_2 < ....$ We also define $B + t = \{s + t | s \in B\}$. The fuzzy Process $\tilde{N}$ is stationary if for any fuzzy Borel sets $B_1, ..., B_k$, $t \geq 0$

$$(\tilde{N}(B_1 + t), ..., \tilde{N}(B_k + t)) \equiv (\tilde{N}(B_1), ..., \tilde{N}(B_k)). \tag{11}$$

**Proposition 3.1** If $\tilde{N}$ is a random fuzzy stationary point process and $E[\tilde{N}(1)] < \infty$, then $E[\tilde{N}(t)] = tE[N(1)], t \geq 0$.

***Proof.*** We have

$$E[\tilde{N}(s+t)]'_\alpha = E[\tilde{N}(s)]'_\alpha + E[\tilde{N}(s+t) - \tilde{N}(s)]'_\alpha = E[\tilde{N}(s)]'_\alpha + E[\tilde{N}(t)]'_\alpha,$$

and

$$E[\tilde{N}(s+t)]''_\alpha = E[\tilde{N}(s)]''_\alpha + E[\tilde{N}(s+t) - \tilde{N}(s)]''_\alpha = E[\tilde{N}(s)]''_\alpha + E[\tilde{N}(t)]''_\alpha,$$

this is a linear equation $f(s+t) = f(s)+f(t), s, t \geq 0$. The only nondecreasing function that satisfies this linear equation is $f(t) = ct$ for some $c$. In our case $c = f(1) = E[\tilde{N}(1)]$,then $E[\tilde{N}(t)]'_\alpha = tE[\tilde{N}(1)]'_\alpha$ and $E[\tilde{N}(t)]''_\alpha = tE[\tilde{N}(1)]''_\alpha$.

Now, assume that $\tilde{N}(t)$ is a random fuzzy delayed renewal process, where the distribution of $\xi_1$ is $G$, and the distribution of $\xi_n, n \geq 2$, is $F$, which has a finite mean $\mu$. The issue is how to select the initial distribution $G$ such that $\tilde{N}$ is stationary. The answer is in following theorem. The following result also shows that the stationary of $\tilde{N}$ is equivalent to the stationary of its forward recurrence time process.

**Theorem 3.1** *The following statements are equivalent:*
*(i) The random fuzzy delayed renewal process $\tilde{N}$ is stationary.*
*(ii) The fuzzy forward recurrence time process $\tilde{B}(t) = T_{\tilde{N}(t)+1} - t$ is stationary.*
*(iii) $E[\tilde{N}(t) = t/\mu]$ for $t \geq 0$.*
*(iv) $G(t) = F_e(t) = 1/\mu \int_0^t [1 - F(s)]ds$.*

***Proof.*** (i)$\Leftrightarrow$ (ii): Using $T_n = \inf\{u : \tilde{N}(u) = n\}$ we have

$$\tilde{B}(t) = T_{\tilde{N}(t)+1} - t = inf\{u - t : \tilde{N}(u) = \tilde{N}(t) + 1\} \tag{12}$$

$$\equiv inf\{t' : \tilde{N}((0, t'] + t) = 1\}.$$

Consequently, the stationary property (10) of $N$ implies $\tilde{B} \equiv \tilde{B}(0), t \geq 0$. Then $\tilde{B}$ is stationary by Remark 3.1.
Conversely, since $\tilde{N}$ counts the number of times $\tilde{B}(t)$ jumps upward,

$$\tilde{N}(A + t) = \sum_{s \in A} 1_{(\tilde{B}(s+t) > \tilde{B}(s+t)-1)}, \tag{13}$$

Therefore, the stationary of $\tilde{B}$ implies $\tilde{N}$ is stationary.
(i)$\Rightarrow$(iii): if $\tilde{N}$ is stationary, Proposition 3.1 ensures $E[\tilde{N}(t)] = tE[\tilde{N}(1)]$. Also, $E[\tilde{N}(1)] = 1/\mu$, therefore $E[\tilde{N}(t)] = t/\mu$.
(iii)$\Rightarrow$(iv): Assume that $E[\tilde{N}(t)] = t/\mu$ and we can show $U'_\alpha \star F_e(t)'_\alpha = (t/\mu)'_\alpha$

and $U''_\alpha \star F_e(t)''_\alpha = (t/\mu)''_\alpha$, so $E[\tilde{N}(t)]'_\alpha = U'_\alpha \star F_e(t)'_\alpha$ and $E[\tilde{N}(t)]''_\alpha = U'_\alpha \star F_e(t)''_\alpha$. In other ways

$$E[\tilde{N}(t)]'_\alpha = \sum_{n=1}^{\infty} G'_\alpha \star F^{n-1}(t)'_\alpha = G'_\alpha \star U(t)'_\alpha \tag{14}$$

and

$$E[\tilde{N}(t)]''_\alpha = \sum_{n=1}^{\infty} G''_\alpha \star F^{n-1}(t)''_\alpha = G''_\alpha \star U(t)''_\alpha \tag{15}$$

where $U(t)'_\alpha = \sum_{n=1}^{\infty}(F^n(t))'_\alpha$ and $U(t)''_\alpha = \sum_{n=1}^{\infty}(F^n(t))''_\alpha$. Therefore, we have $U'_\alpha \star F_e(t)'_\alpha = G'_\alpha \star U(t)'_\alpha$ and $U''_\alpha \star F_e(t)''_\alpha = G''_\alpha \star U(t)''_\alpha$. Taking the Laplace transform of this equality yields

$$\hat{U}(s)'_\alpha \star \hat{F}_e(s)'_\alpha = \hat{G}(s)'_\alpha \star \hat{U}(s)'_\alpha \tag{16}$$

and

$$\hat{U}(s)''_\alpha \star \hat{F}_e(s)''_\alpha = \hat{G}(s)''_\alpha \star \hat{U}(s)''_\alpha \tag{17}$$

Using that $\hat{U}(s)'_\alpha = 1/(1-\hat{F}(s)'_\alpha)$ and $\hat{U}(s)''_\alpha = 1/(1-\hat{F}(s)''_\alpha)$ is positive, (18) and (19) yield $\hat{F}_e(s)'_\alpha = \hat{G}(s)'_\alpha$ and $\hat{F}_e(s)''_\alpha = \hat{G}(s)''_\alpha$. Then we obtain $G'_\alpha = (F_e)'_\alpha$ and $G''_\alpha = (F_e)''_\alpha$.
(iv)$\Rightarrow$(ii): We have

$$(P\{\tilde{B}(t) > x\})'_\alpha = 1 - \tilde{G}(t+x)'_\alpha + \int_0^t [1 - F(t+x-s)]'_\alpha dV(s), \tag{18}$$

and

$$(P\{\tilde{B}(t) > x\})''_\alpha = 1 - \tilde{G}(t+x)''_\alpha + \int_0^t [1 - F(t+x-s)]''_\alpha dV(s), \tag{19}$$

where $V(t) = E[\tilde{N}(t)] = G \star U(t)$. Now, the assumption $G = F_e$ yields

$$(V(t))'_\alpha = \tilde{G}'_\alpha \star U(t)'_\alpha = U'_\alpha \star \tilde{G}(t)'_\alpha = U'_\alpha \star (F_e)'_\alpha = (t/\mu)'_\alpha, \tag{20}$$

and

$$(V(t))''_\alpha = \tilde{G}''_\alpha \star U(t)''_\alpha = U''_\alpha \star \tilde{G}(t)''_\alpha = U''_\alpha \star (F_e)''_\alpha = (t/\mu)''_\alpha. \tag{21}$$

Using (18) and (19), along with a change of variable in the integral, we have

$$(P\{\tilde{B}(t) > x\})'_\alpha = 1 - \tilde{G}(t+x)'_\alpha + F_e(t+x)]'_\alpha - F_e(x)]'_\alpha, \tag{22}$$

and

$$(P\{\tilde{B}(t) > x\})''_\alpha = 1 - \tilde{G}(t+x)''_\alpha + F_e(t+x)]''_\alpha - F_e(x)]''_\alpha. \tag{23}$$

Since $G = F_e$, based on these expression, we conclude that $(P\{\tilde{B}(t) > x\})'_\alpha = 1 - F_e(x)]'_\alpha$ and $(P\{\tilde{B}(t) > x\})''_\alpha = 1 - F_e(x)]''_\alpha, t \geq 0$. Thus, the distribution of $\tilde{B}(t)$ is dependent of $t$. This condition is sufficient for $\tilde{B}(t)$ to be stationary$\bullet$

**Corollary** The random fuzzy renewal process $\tilde{N}(t)$ with no delay, and whose inter renewal times have a finite mean, is stationary if and only if it is a random fuzzy poisson process.

# 4 Conclusion

We have shown that a random fuzzy delayed renewal process is stationary if the fuzzy forward recurrence time process be stationary. Also, it can be stationary under conditions stated in Theorem 1. Thus, one recommendation for future research is to consider the stationary of random fuzzy renewal by the fuzzy backward recurrence time $\tilde{A}(t) = t - T_{\tilde{N}(t)}$. It seems reasonable that the fuzzy backward recurrence time would also be stationary. But, this is not true, since the distribution of $\tilde{A}(t)$ is not independent of $t$. However, possibly there is stationary in spacial case.

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