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SOME RESULTS ON JANOWSKI CLOSE TO CONVEX MAPPINGS

Melike Aydoğan

Department of Mathematics
Isik University- Sile Kampusu
Mesrutiyet Koyu Universite Sokak, D Kapi No:2
34980, Istanbul, Turkey

Abstract

Let $J_C(A,B)$ denote the class of functions $\phi(z) = z + \sum_{k=2}^{\infty} a_k z^k$ are analytic in the open unit disc D such that

$$\frac{z\phi'(z)}{s(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1 \tag{1}$$

where $s(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is convex in D. In this paper we determine the coefficient estimates distortion theorems for this class.

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1 Introduction

Let $H(D)$ be the linear space of all analytic functions defined in the open unit disc D. Let $w(z) = \sum_{k=1}^{\infty} c_k z^k$ be an analytic function in the open unit disc and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in D$.
Let C denote the class of functions such as

$$s(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{2}$$

analytic and convex in D.
Let J_C denote the class of functions such as

$$\psi(z) = \int_0^z \frac{s(z)}{z} \tag{3}$$

is Janowski and convex in D.
Let $J_C(A,B)$ denote the class of functions such as

$$\phi(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{4}$$

analytic in D and satisfying the conditions;

$$\frac{z\phi'(z)}{s(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in D, s(z) \in C \tag{5}$$

If we use the definition of subordination principle for $\phi(z) \in J_C(A,B)$ if and only if $\phi(z)$ can be represented in the form:

$$\frac{z\phi'(z)}{s(z)} = \frac{1+Aw(z)}{1+Bw(z)}, w(z) \in H(D), -1 \leq B < A \leq 1, z \in D \tag{6}$$

We study the $J_C(A,B)$ and obtain coefficient estimates, distortion theorems.

2 Some Preliminary Lemmas

We need the following lemmas.

Lemma 2.1 *Let* $\frac{z\phi'(z)}{s(z)} = p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ *then* $|p_n| \leq (A-B)$, $n \geq 1$. *The bounds are sharp, being attained for the functions*

$$p_n(z) = \frac{1+A\delta z^n}{1+B\delta z^n}, |\delta| = 1 \tag{7}$$

[Goel and Mehrok].

Lemma 2.2 *If* $w(z) \in H(D)$ *then for* $|z| = r < 1$

$$|zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1-r^2} \tag{8}$$

[Singh and Goel].

Lemma 2.3 *Let* $p(z) = \frac{1+Bw(z)}{1+Aw(z)}$, $w(z) \in H(D)$, *then for* $|z| = r < 1$

$$Re[Ap(z) + \frac{B}{p(z)}] + \frac{r^2 |Ap(z) - B|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|} \leq$$

$$\frac{AB(A+B)r^2 - 4ABr + (A+B)}{(1-Ar)(1-Br)}; R_1 \leq R_0,$$

$$\frac{2}{(1-r^2)}(1-ABr^2)-[(1-A)(1-B)(1+Ar^2)(1+Br^2)]^{\frac{1}{2}}; R_1 \geq R_0.$$

where $A \neq 1$, $R_1 = \frac{1-Br}{1-Ar}$ *and*

$$R_0^2 = \frac{(1-B)(1+Br^2)}{(1-A)(1+Ar^2)}$$

The bounds are sharp.[Goel and Mehrok].

3 Coefficient Inequalities

Theorem 3.1 *If* $\phi(z) \in J_C(A,B)$ *then,*

$$|a_n| \leq \frac{1}{n} + \frac{(n-1)(A-B)}{n}, n \geq 2 \tag{9}$$

The bounds are sharp.

Proof: Using (1.2) and (1.4) in (2.1) we get;

$$z(1+\sum_{k=2}^{\infty} ka_k z^{k-1}) = (z+\sum_{k=2}^{\infty} b_k z^k)(1+\sum_{k=1}^{\infty} p_k z^k) \tag{10}$$

Equating the coefficient of z^n in (3.2) we have;

$$na_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + ... + p_{n-1} \tag{11}$$

Therefore using (3.2),

$$n\,|a_n| \leq |b_n| + (A-B)[|b_{n-1}| + |b_{n-2}| + ... + |b_2| + 1] \tag{12}$$

Also it is well known that $|b_n| \leq 1$, $n \geq 2$. Hence $|a_n| \leq \frac{1}{n} + \frac{(n-1)(A-B)}{n}$ For $n = 2$ equality signs in (3.1) holds for the function $\phi(z)$

$$\phi_n'(z) = \frac{1}{(1-\delta_1 z)}\frac{1+A\delta_2 z^{n-1}}{1+B\delta_2 z^{n-1}}, |\delta_1| = 1, |\delta_2| = 1 \tag{13}$$

On putting $A = 1$, $B = -1$ in above theorem , we get the following result due to Grawod and Thomas.

Corollary 3.2 *Let* $\phi(z) \in J$, *then* $|a_n| \leq 2 - \frac{1}{n}$

4 Main Results

Theorem 4.1 *If $\phi(z) \in J_C(A,B)$ then for $|z| = r$, $0 < r < 1$*

$$\frac{1-Ar}{(1-Br)^{\frac{2B-A}{B}}} \leq |\phi'(z)| \leq \frac{1+Ar}{(1-Br)^{\frac{2B-A}{B}}}; B \neq 0, \tag{14}$$

$$\frac{e^{-Ar}(1-Ar)}{r(1-Br)} \leq |\phi'(z)| \leq \frac{e^{Ar}(1+Ar)}{r(1-Br)} B = 0. \tag{15}$$

$$\int_0^r \frac{1-At}{(1-Bt)^{\frac{2B-A}{B}}} dt \leq |\phi(z)| \leq \int_0^r \frac{1+At}{(1-Bt)^{\frac{2B-A}{B}}}; B \neq 0, \tag{16}$$

$$\int_0^r \frac{e^{-At}(1-At)}{t(1-Bt)} dt \leq |\phi(z)| \leq \int_0^r \frac{e^{At}(1+At)}{t(1-Bt)} dt; B = 0. \tag{17}$$

Estimates are sharp.

Proof: Let $s(z) = z + b_2 z^2 + ...$ analytic and convex;

$$Re(1 + z\frac{s''(z)}{s(z)}) > 0$$

If $\frac{\phi'(z)}{s'(z)} > 0$

$$Re\frac{\phi'(z)}{s'(z)} = \frac{1+A\phi(z)}{1+B\phi(z)}$$

$$z\frac{s'(z)}{s(z)} = \frac{1+A\phi(z)}{1+B\phi(z)}$$

Let $\psi(z)$ Janowski and convex $\in J_C$; $\psi'(z) = \frac{s(z)}{z}$,

$$\log \psi'(z) = \log s(z) - \log z$$

$$\frac{\psi''(z)}{\psi'(z)} = \frac{1}{z} + \frac{s'(z)}{s(z)} \Rightarrow z\frac{\psi''(z)}{\psi(z)} = -1 + z\frac{s'(z)}{s(z)} \Rightarrow$$

$$1 + z\frac{\psi''(z)}{\psi'(z)} = z\frac{s'(z)}{s(z)} \Rightarrow \psi(z) is convex$$

$$1 + z\frac{\psi''(z)}{\psi'(z)} = z\frac{s'(z)}{s(z)} = \frac{1+A\phi(z)}{1+B\phi(z)}$$

Therefore we can write thw following inequallities;

$$r(1-Br)^{\frac{A-B}{B}} \leq |s(z)| \leq r(1+Br)^{\frac{A-B}{B}}; B \neq 0, \tag{18}$$

$$e^{-Ar} \leq |s(z)| \leq e^{Ar}; B = 0. \tag{19}$$

And from the subordination principle we can write ;

$$z\frac{s'(z)}{s(z)} = \frac{1+Aw(z)}{1+Bw(z)} \Leftrightarrow z\frac{s'(z)}{s(z)} \prec \frac{1+Az}{1+Bz}$$

$$\left|z\frac{s'(z)}{s(z)} - \frac{1-ABr^2}{1-Br^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}$$

$$\frac{1-Ar}{1-Br} \leq \left|z\frac{s'(z)}{s(z)}\right| \leq \frac{1+Ar}{1-Br}$$

$$\frac{1}{r}\left|s(z)\right|\frac{1-Ar}{1-Br} \leq \left|s'(z)\right| \leq \frac{1}{r}\left|s(z)\right|\frac{1+Ar}{1-Br}$$

$$\left|z\frac{\phi'(z)}{s(z)} - \frac{1-ABr^2}{1-Br^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}$$

$$\frac{1-Ar}{1+Br} \leq \left|\frac{z\phi'(z)}{s(z)}\right| \leq \frac{1+Ar}{1-Br}$$

And with simple calculations we can take the result easily.

Lemma 4.2 *Let $s(z)$ be an element of $J_C(A,B)$ then ;*

$$(1-Ar)(1-Br)^{\frac{A-2B}{B}} \leq |s'(z)| \leq (1+Ar)(1+Br)^{\frac{A-2B}{B}}; B \neq 0, \tag{20}$$

$$(1-Ar)e^{-Ar} \leq |s'(z)| \leq (1+Ar)e^{Ar}; B = 0. \tag{21}$$

Proof: Using the definition of the class $J_C(A,B)$ and subordination principle , then we write;

$$\left|\frac{s'(z)}{\phi'(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}; B \neq 0, \tag{22}$$

$$\left|\frac{s'(z)}{\phi'(z)} - 1\right| \leq Ar; B = 0. \tag{23}$$

These inequalities can be written in the following form;

$$\frac{1-Ar}{1-Br} \leq \frac{s'(z)}{\phi'(z)} \leq \frac{1+Ar}{1+Br}; B \neq 0, \tag{24}$$

$$(1-Ar) \leq \frac{s'(z)}{\phi'(z)} \leq (1+Ar); B = 0. \tag{25}$$

then we have the result.

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