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# Some remarks on commutative and pointed pseudo-CI algebras

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**Abstract**

In this paper, the concepts of branchwise commutative pseudo-CI algebras and pointed pseudo-CI algebras are introduced and some of their properties are investigated.

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## 1 Introduction

In 1966, Y. Imai and K. Iséki ([9]) introduced the notion of a BCK algebra. There exist several generalizations of BCK algebras, such as BCI algebras

([10]), BCH algebras ([8]), BE algebras ([11]), CI algebras ([12]), etc. In 2001, G. Georgescu and A. Iorgulescu ([7]) introduced pseudo-BCK algebras as an extension of BCK algebras. In 2008, W. A. Dudek and Y. B. Jun ([6]) defined pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCI algebras. G. Dymek studied $p$-semisimple pseudo-BCI algebras ([4]) and periodic pseudo-BCI algebras ([5]). R. A. Borzooei et al. ([1]) defined pseudo-BE algebras which are a generalization of BE algebras. A. Walendziak introduced pseudo-BCH algebras and then investigated ideals in such algebras ([14]–[16]). Recently, A. Rezaei et al. defined the class of pseudo-CI algebras and studied some of its subclasses ([13]).

In this paper, we introduce and study the concept of a branchwise commutative pseudo-CI algebra. We also introduce pointed pseudo-CI algebras and investigate some of their properties.

## 2 Preliminary Notes

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.
Recall that a *CI algebra* ([12]) is an algebra $(X; \rightarrow, 1)$ of type $(2, 0)$ satisfying the following axioms:

(CI1) $x \rightarrow x = 1$,
(CI2) $1 \rightarrow x = x$,
(CI3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

A CI-algebra $(X; \rightarrow, 1)$ is said to be a *BE-algebra* ([11]) if

(BE) $x \rightarrow 1 = 1$

holds in $(X; \rightarrow, 1)$.

An algebra $\mathfrak{X} = (X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-BCI algebra* if it satisfies the following axioms:

($\mathrm{I}_1$) $(x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z)) = 1$,
($\mathrm{I}_2$) $(x \rightsquigarrow y) \rightarrow ((y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)) = 1$,
($\mathrm{I}_3$) $x \rightarrow ((x \rightarrow y) \rightsquigarrow y) = 1$ and $x \rightsquigarrow ((x \rightsquigarrow y) \rightarrow y) = 1$,
($\mathrm{I}_4$) $x \rightarrow y = y \rightsquigarrow x = 1 \Longrightarrow x = y$,
($\mathrm{I}_5$) $x \rightarrow x = x \rightsquigarrow x = 1$,
($\mathrm{I}_6$) $x \rightarrow y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

A pseudo-BCI algebra $\mathfrak{X}$ satisfying condition

($\mathrm{I}_7$) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$

is said to be a *pseudo-BCK algebra* ([7]).

**Definition 2.1.** ([13]) An algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-CI algebra* if it satisfies $(I_5)$, $(I_6)$ and the following axioms:

$(I_8)$ $1 \rightarrow x = 1 \rightsquigarrow x = x$,
$(I_9)$ $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$.

**Remark 2.2.** If $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo-CI algebra satisfying $x \rightarrow y = x \rightsquigarrow y$, for all $x, y \in X$, then $(X; \rightarrow, 1)$ is a CI-algebra.

A pseudo-CI algebra $(X; \rightarrow, \rightsquigarrow, 1)$ is called:
(i) a *pseudo-BCH algebra* ([14]) if it verifies $(I_4)$,
(ii) a *pseudo-BE algebra* ([1]) if it verifies $(I_7)$.

**Remark 2.3.** Since every pseudo-BCI algebra satisfies $(I_4)$–$(I_6)$, $(I_8)$ and $(I_9)$, pseudo-BCI algebras are contained in the class of pseudo-BCH algebras.

Denote by **psBCK**, **psBCI**, **psCI**, **psBE** and **psBCH** the classes of pseudo-BCK, pseudo-BCI, pseudo-CI, pseudo-BE and pseudo-BCH algebras, respectively. By definition and Remark 2.3,

$$\mathbf{psBCK} \subset \mathbf{psBCI} \subset \mathbf{psBCH} \subset \mathbf{psCI} \quad \text{and} \quad \mathbf{psBE} \subset \mathbf{psCI}.$$

**Remark 2.4.** The class of all pseudo-CI algebras is a quasi-variety. Therefore, if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two pseudo-CI algebras, then the direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is also a pseudo-CI algebra.

**Example 2.5.** (1) Let $X_1 = \{1, a, b, c, d\}$. Define the operations $\rightarrow_1$ and $\rightsquigarrow_1$ on $X_1$ by the following tables:

| $\rightarrow_1$ | 1 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $c$ | $c$ | 1 |
| $b$ | 1 | $d$ | 1 | 1 | $d$ |
| $c$ | 1 | $d$ | 1 | 1 | $d$ |
| $d$ | 1 | 1 | $c$ | $c$ | 1 |

and

| $\rightsquigarrow_1$ | 1 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $b$ | $c$ | 1 |
| $b$ | 1 | $d$ | 1 | 1 | $d$ |
| $c$ | 1 | $d$ | 1 | 1 | $d$ |
| $d$ | 1 | 1 | $b$ | $c$ | 1 |

We can observe that $\mathfrak{X}_1 = (X_1; \rightarrow_1, \rightsquigarrow_1, 1)$ is a pseudo-CI algebra, which is not a pseudo-BCH algebra since $b \neq c$ and $b \rightarrow_1 c = c \rightsquigarrow_1 b = 1$.

(2) Let $X_2 = \{1, a, b, c, d\}$. Define the operations $\rightarrow_2$ and $\rightsquigarrow_2$ on $X_2$ by the following tables:

| $\rightarrow_2$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | $a$ | 1 | $d$ | $e$ | $b$ | $c$ |
| $b$ | $b$ | $c$ | 1 | $a$ | $e$ | $d$ |
| $c$ | $d$ | $e$ | $a$ | 1 | $c$ | $b$ |
| $d$ | $c$ | $b$ | $e$ | $d$ | 1 | $a$ |
| $e$ | $e$ | $d$ | $c$ | $b$ | $a$ | 1 |

and

| $\rightsquigarrow_2$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | $a$ | 1 | $c$ | $b$ | $e$ | $d$ |
| $b$ | $b$ | $d$ | 1 | $e$ | $a$ | $c$ |
| $c$ | $d$ | $b$ | $e$ | 1 | $c$ | $a$ |
| $d$ | $c$ | $e$ | $a$ | $d$ | 1 | $b$ |
| $e$ | $e$ | $c$ | $d$ | $a$ | $b$ | 1 |

Then $\mathfrak{X}_2 = (X_2; \rightarrow_2, \rightsquigarrow_2, 1)$ is a pseudo-CI algebra, which is not a pseudo-BE algebra since $a \rightarrow_2 1 = a \neq 1$.

(3) The direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a pseudo-CI algebra, which is neither a pseudo-BCH algebra nor a pseudo-BE algebra.

Let $(X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-CI algebra. We define the binary relation $\preceq$ by: for all $x, y \in X$,

$$x \preceq y \Longleftrightarrow x \rightarrow y = 1 \Longleftrightarrow x \rightsquigarrow y = 1.$$

We note that $\preceq$ is reflexive by ($I_5$). From [13] we have (for all $x, y, z \in X$):

(a1) $1 \preceq x \Longrightarrow x = 1$,
(a2) $x \preceq y \rightarrow z \Longleftrightarrow y \preceq x \rightsquigarrow z$,
(a3) $x \rightarrow 1 = x \rightsquigarrow 1$,
(a4) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightarrow 1)$,
(a5) $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightsquigarrow 1)$,
(a6) $x \preceq y \Longrightarrow x \rightarrow 1 = y \rightarrow 1 = x \rightsquigarrow 1 = y \rightsquigarrow 1$,
(a7) $y \preceq (y \rightarrow x) \rightsquigarrow x$ and $y \preceq (y \rightsquigarrow x) \rightarrow x$.

**Proposition 2.6.** *Let $\mathfrak{X} = (X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-CI algebra. Then:*

(i) *If $a, x, y \in X$, $x \preceq a$ and $y \preceq a$, then $x \rightarrow y \preceq 1$ and $x \rightsquigarrow y \preceq 1$.*
(ii) *If $x, y \in X$, $x \preceq 1$, then $y \preceq x \rightarrow y$ and $y \preceq x \rightsquigarrow y$.*

*Proof.* (i) Let $x \preceq a$ and $y \preceq a$. By (a4) and (a6), $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightarrow 1) = (a \rightarrow 1) \rightsquigarrow (a \rightarrow 1) = 1$, and hence $x \rightarrow y \preceq 1$. Similarly, $x \rightsquigarrow y \preceq 1$.

(ii) Since $x \preceq 1$ and $1 = y \rightarrow y$, we have $x \preceq y \rightarrow y$. Applying (a2), we get $y \preceq x \rightsquigarrow y$. Similarly, $y \preceq x \rightarrow y$. □

A *pseudo-CI algebra with condition (A)*, or a *pseudo-CI(A) algebra* for short, is a pseudo-CI algebra $(X; \rightarrow, \rightsquigarrow, 1)$ satisfying the following condition:

(A) for all $x, y, z \in X$, if $x \preceq y$, then $y \rightarrow z \preceq x \rightarrow z$ and $y \rightsquigarrow z \preceq x \rightsquigarrow z$.

Clearly, every pseudo-BCI algebra satisfies (A). It is easy to see that the pseudo-CI algebra from Example 2.5 (1) also satisfies (A). The following example shows that there are pseudo-CI algebras, which do not satisfy (A).

**Example 2.7.** Let $X = \{1, a, b, c, d, e, f\}$. We define the binary operations $\rightarrow$ and $\rightsquigarrow$ on $X$ as follows:

| $\rightarrow$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $a$ | 1 | 1 | $b$ | $b$ | $d$ | $e$ | $f$ |
| $b$ | 1 | $a$ | 1 | $c$ | $d$ | $e$ | $f$ |
| $c$ | 1 | 1 | 1 | 1 | $d$ | $e$ | $f$ |
| $d$ | 1 | $a$ | $b$ | $c$ | 1 | 1 | $f$ |
| $e$ | 1 | $a$ | $b$ | $c$ | $e$ | 1 | 1 |
| $f$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | 1 |

and

| $\rightsquigarrow$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $a$ | 1 | 1 | $b$ | $c$ | $d$ | $e$ | $f$ |
| $b$ | 1 | $a$ | 1 | $a$ | $d$ | $e$ | $f$ |
| $c$ | 1 | 1 | 1 | 1 | $d$ | $e$ | $f$ |
| $d$ | 1 | $a$ | $b$ | $c$ | 1 | 1 | $f$ |
| $e$ | 1 | $a$ | $b$ | $c$ | $e$ | 1 | 1 |
| $f$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ | 1 |

Routine calculations show that $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a pseudo-CI algebra. $\mathfrak{X}$ does not satisfy (A). Indeed, we have $d \preceq e$ but $e \to f = 1 \not\preceq f = d \to f$.

Let $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-CI algebra. An element $a \in X$ is said to be an *atom* of $\mathfrak{X}$ (see [13]) if for any $x \in X$, $a \preceq x$ implies $a = x$. Let $\mathrm{A}(\mathfrak{X})$ denote the set of all atoms of $\mathfrak{X}$. By (a1), $1 \in \mathrm{A}(\mathfrak{X})$, so $\mathrm{A}(\mathfrak{X}) \neq \emptyset$. From [13] it follows that

$$a \in \mathrm{A}(\mathfrak{X}) \text{ if and only if } a = (a \to x) \rightsquigarrow x \text{ for all } x \in X.$$

Remark that if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are pseudo-CI algebras from Example 2.5 (1) and (2), respectively, then $\mathrm{A}(\mathfrak{X}_1) = \{1\}$ and $\mathrm{A}(\mathfrak{X}_2) = X_2$.

# 3 Branchwise commutativity

Let $\mathfrak{X}$ be a pseudo-CI algebra. For any $a \in X$ we define a subset $\mathrm{V}(a)$ of $X$ as follows

$$\mathrm{V}(a) = \{x \in X : x \preceq a\}.$$

Remark that $\mathrm{V}(a) \neq \emptyset$, since $a \preceq a$ gives $a \in \mathrm{V}(a)$.

If $a$ is an atom of $\mathfrak{X}$, then the set $\mathrm{V}(a)$ is called a *branch* of $\mathfrak{X}$ determined by element $a$.

The pseudo-CI algebras from Examples 2.5 (1) and 2.7 have only one branch; the algebra given in Example 2.5 (2) has the following branches: $\{1\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$ and $\{f\}$.

A pseudo-CI algebra $\mathfrak{X}$ is called *commutative* ([13]) if for all $x, y, z \in X$, it satisfies the following identities:

$$(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x \tag{3.1}$$

$$(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x. \tag{3.2}$$

From [13] it follows that any commutative pseudo-CI algebra is a pseudo-BCK algebra (and also pseudo-BE algebra). Hence we obtain

**Proposition 3.1.** *Commutative pseudo-CI algebras coincide with commutative pseudo-BE algebras, with commutative pseudo-BCI algebras and with commutative pseudo-BCK algebras.*

In [3], G. Dymek introduced the notion of branchwise commutative pseudo-BCI algebras. A. Walendziak ([17]) introduced and studied branchwise commutative pseudo-BCH algebras. Following [3] and [17], we say that a pseudo-CI algebra $\mathfrak{X}$ is *branchwise commutative* if identities (3.1) and (3.2) hold for $x$ and $y$ belonging to the same branch. Clearly, any commutative pseudo-CI algebra is branchwise commutative. Note that the pseudo-CI algebra from Example 2.5 (2) is branchwise commutative, but it is not commutative since $(a \rightarrow_2 b) \rightsquigarrow_2 b = a \neq b = (b \rightarrow_2 a) \rightsquigarrow_2 a$.

**Proposition 3.2.** *Let $\mathfrak{X}$ be a pseudo-CI algebra and $a \in X$. Then* $(\mathrm{V}(a); \preceq)$ *is a poset.*

*Proof.* Obviously, $\preceq$ is reflexive. Observe that $\preceq$ is also anti-symmetric. Indeed, let $x, y \in \mathrm{V}(a)$. Suppose that $x \preceq y$ and $y \preceq x$. Then $x \rightarrow y = y \rightarrow x = 1$. By ($\mathrm{I}_8$) and branchwise commutativity,

$$x = 1 \rightsquigarrow x = (y \rightarrow x) \rightsquigarrow x = (x \rightarrow y) \rightsquigarrow y = 1 \rightsquigarrow y = y.$$

Now we prove that $\preceq$ is transitive. Let $x, y, z \in \mathrm{V}(a)$ and assume that $x \preceq y$ and $y \preceq z$. Then $x \rightarrow y = y \rightarrow z = 1$. By Proposition 2.6 (i), $z \rightarrow y \preceq 1$. We have

$$\begin{aligned} x \rightarrow z &= (y \rightarrow z) \rightsquigarrow (x \rightarrow z) && [\text{by } (\mathrm{I}_8)] \\ &= x \rightarrow ((y \rightarrow z) \rightsquigarrow z) && [\text{by } (\mathrm{I}_9)] \\ &= x \rightarrow ((z \rightarrow y) \rightsquigarrow y) && [\text{by branchwise commutativity}] \\ &= (z \rightarrow y) \rightsquigarrow (x \rightarrow y) && [\text{by } (\mathrm{I}_9)] \\ &= (z \rightarrow y) \rightsquigarrow 1 = 1 && [\text{since } z \rightarrow y \preceq 1]. \end{aligned}$$

Hence $x \preceq z$, and therefore $\preceq$ is transitive. Consequently, $(\mathrm{V}(a); \preceq)$ is a poset. □

**Theorem 3.3.** *Let $\mathfrak{X}$ be a pseudo-CI(A) algebra. The following statements are equivalent:*
(a) *$\mathfrak{X}$ is branchwise commutative.*
(b) *Each branch of $\mathfrak{X}$ is a semilattice with respect to the join $\vee$ defined by*

$$x \vee y = (x \rightarrow y) \rightsquigarrow y = (x \rightsquigarrow y) \rightarrow y.$$

*Proof.* (a) $\Longrightarrow$ (b): Suppose that $\mathfrak{X}$ is branchwise commutative. Let $a \in \mathrm{A}(\mathfrak{X})$. By Proposition 3.2, $(\mathrm{V}(a); \preceq)$ is a poset. Let $x, y \in \mathrm{V}(a)$. From Proposition 2.6 (i) we conclude that $x \rightarrow y \preceq 1$, $x \rightsquigarrow y \preceq 1$, $y \rightarrow x \preceq 1$, and $y \rightsquigarrow x \preceq 1$. Applying Proposition 2.6 (ii), we get $x, y \preceq (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$ and $x, y \preceq (x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$. Observe that $(x \rightarrow y) \rightsquigarrow y \in \mathrm{V}(a)$. Since $x \preceq a$, using (A), we see that $a \rightarrow y \preceq x \rightarrow y$, and hence

$$(x \rightarrow y) \rightsquigarrow y \preceq (a \rightarrow y) \rightsquigarrow y. \tag{3.3}$$

By (a7), $a \preceq (a \rightarrow y) \rightsquigarrow y$. From this, since $a \in \mathrm{A}(\mathfrak{X})$, we have $a = (a \rightarrow y) \rightsquigarrow y$. Therefore, by (3.3), $(x \rightarrow y) \rightsquigarrow y \preceq a$, and similarly, $(x \rightsquigarrow y) \rightarrow y \preceq a$. Then $(x \rightarrow y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \rightarrow y$ belong to $\mathrm{V}(a)$, and they are upper bounds of $\{x, y\}$. Now we show that $(x \rightarrow y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \rightarrow y$

are both the least upper bounds of $\{x, y\}$. Let $z \in \mathrm{V}(a)$ be another upper bound of $\{x, y\}$. Therefore, $x \preceq z$ and $y \preceq z$. By branchwise commutativity, $(z \to y) \rightsquigarrow y = (y \to z) \rightsquigarrow z = 1 \rightsquigarrow z = z$, and similarly, $(z \rightsquigarrow y) \to y = z$. Since $x \preceq z$, applying (A) we deduce that $(x \to y) \rightsquigarrow y \preceq (z \to y) \rightsquigarrow y = z$ and $(x \rightsquigarrow y) \to y \preceq (z \rightsquigarrow y) \to y = z$, that is, $(x \to y) \rightsquigarrow y \preceq z$ and $(x \rightsquigarrow y) \to y \preceq z$. Thus $(x \to y) \rightsquigarrow y$ and $(x \rightsquigarrow y) \to y$ are both the least upper bounds of $\{x, y\}$. Then $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$ and $\mathrm{V}(a)$ is a semilattice with respect to $\vee$.

(b) $\Longrightarrow$ (a): Let $x$ and $y$ belong to the same branch. By assumption, $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$. Since $x \vee y = y \vee x$, we obtain identities (3.1) and (3.2). Thus $\mathfrak{X}$ is branchwise commutative. □

Let $\mathfrak{X}$ be a commutative pseudo-CI algebra. Then $\mathfrak{X}$ is a pseudo-BCK algebra, and therefore it satisfies (A) and has only one branch. Consequently, from Theorem 3.3 we obtain

**Corollary 3.4.** *Any commutative pseudo-*CI *algebra is a join-semilattice with respect to* $\preceq$.

Since every pseudo-BE algebra is a pseudo-CI algebra, we have

**Corollary 3.5.** *Any commutative pseudo-*BE *algebra is a join-semilattice with respect to* $\preceq$.

## 4 Pointed pseudo-CI algebras

**Definition 4.1.** A pseudo-CI algebra $(X; \to, \rightsquigarrow, 1)$ with a constant element $a$ (which can denoted any element) is called a *pointed pseudo-CI algebra* and it is denoted by $(X; \to, \rightsquigarrow, a, 1)$.

For any pointed pseudo-CI algebra $(X; \to, \rightsquigarrow, a, 1)$, for all $x \in X$, define two negations relative to $a$:

$$x^{-_a} := x \to a, \quad x^{\sim_a} := x \rightsquigarrow a,$$

**Proposition 4.2.** *Let* $(X; \to, \rightsquigarrow, a, 1)$ *be a pointed pseudo-CI algebra. Then the following hold:*

(p1) $a^{-_a} = a^{\sim_a} = 1$ *and* $1^{-_a} = 1^{\sim_a} = a$,
(p2) $x^{-_1} = x^{\sim_1}$,
(p3) *if* $x \preceq a$, *then* $x^{-_1} = x^{\sim_1} = a^{\sim_1} = a^{-_1}$,
(p4) $x \preceq x^{-_a \sim_a}$, $x \preceq x^{\sim_a -_a}$,
(p5) *if* $x \to y = a$ *or* $x \rightsquigarrow y = a$, *then* $y^{\sim_a} = x^{-_1} = x^{\sim_1} = y^{-_a}$,
(p6) $x \preceq y^{-_a} \Longleftrightarrow y \preceq x^{\sim_a}$,
(p7) $x^{-_a} = 1 \Longleftrightarrow x^{\sim_a} = 1$.

*Proof.* (p1) We have $a \to a = a \rightsquigarrow a = 1$ and $1 \to a = 1 \rightsquigarrow a = a$. Hence (p1) holds.

(p2) Follows from (a3).

(p3) Obviously, by (a6).

(p4) Follows immediately from (a7).

(p5) Let $x \to y = a$. Applying ($I_9$) and ($I_5$) we obtain $y^{\sim_a} = y \rightsquigarrow a = y \rightsquigarrow (x \to y) = x \to (y \rightsquigarrow y) = x \to 1 = x^{-_1}$. Similarly, $y^{-_a} = x^{\sim_1}$. Then $y^{\sim_a} = x^{-_1} = x^{\sim_1} = y^{-_a}$ by (p2). If $x \rightsquigarrow y = a$, then the proof is similar.

(p6) Clearly, by (a2).

(p7) Follows from ($I_6$). □

**Definition 4.3.** A pointed pseudo-CI algebra $(X; \to, \rightsquigarrow, a, 1)$ is called *a-good* if $x^{-_a\sim_a} = x^{\sim_a-_a}$ for all $x \in X$.

**Remark 4.4.** By Proposition 4.2 (p2), every pointed pseudo-CI algebra $\mathfrak{X}' = (X; \to, \rightsquigarrow, 1, 1)$ is 1-good.

**Example 4.5.** (1) Let $(X_1; \to, \rightsquigarrow, 1)$ be the pseudo-CI algebra given in Example 2.5 (1). Then the pointed pseudo-CI algebra $(X_1; \to_1, \rightsquigarrow_1, x, 1)$ is $x$-good for all $x \in X_1$.
(2) Consider the pseudo-CI algebra $\mathfrak{X}$ from Example 2.7. Observe that the pointed pseudo-CI algebra $(X; \to, \rightsquigarrow, c, 1)$ is not $c$-good. Indeed, $(a \to c) \rightsquigarrow c = a \neq 1 = (a \rightsquigarrow c) \to c$.

If $(X; \to, \rightsquigarrow, 1)$ is a commutative pseudo-BE algebra, then

$$(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y \tag{4.1}$$

for all $x, y \in X$ (see Corollary 1.2 of [2]). By Proposition 3.1, every commutative pseudo-CI algebra satisfies (4.1). From this we have

**Corollary 4.6.** *Every pointed commutative pseudo-CI algebra* $(X; \to, \rightsquigarrow, a, 1)$ *is a-good.*

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