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Some Properties of Binormal and Complex Symmetric Operators

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Abstract

In this paper, first, we describe the conditions that a binormal operator becomes a normal operator and an n-binormal becomes an n-normal. we also give some properties of binormal, n-binormal and n-normal. Second, we solve the problem that binormal is not closed under addition. Third, we describe necessary and sufficient conditions that a skew complex symmetric operator becomes a binormal opetrator.

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1 Introduction

We denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle .,. \rangle$ and by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A conjugation on $\mathcal{H}$ is an anti-linear operator $C : \mathcal{H} \to \mathcal{H}$, which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric, if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be skew complex symmetric, if there exists a conjugation C on $\mathcal{H}$ such that $T = -CT^*C$. This terminology is due to the fact that T is complex symmetric if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of appropriate dimension(see[1]). All normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators and Volterra integration operators are included in the class of complex symmetric operators. Garcia initiated the study of complex symmetric operators in [1], [2] and [3], researchers have made progress in the last decade.

An operator is said to be normal, if T commutes with T^*. If T commutes with T^*T, we called T is quasinormal. An operator is said to be binormal, if T^*T and

TT^* commute. An operator $T \in \mathcal{L}(\mathcal{H})$ is called n-normal, if $T^*T^n = T^nT^*$. An operator $T \in \mathcal{L}(\mathcal{H})$ is n-binormal, if T^*T^n commutes with T^nT^* or $[T^*T^n, T^nT^*] = 0$. The study of binormal operators was initiated by Campbell [5] in 1972. Campbell realizes in his second paper [4] have already used the term binormal for an entirely different condition. Campbell stated that usage of term was not current, so he continued his use of binormal to describe the definition we use here. Unfortunately, both definitions are still used in [6], [7]. In particular, Garcia and wogen have shown that every binormal operator [in the sense of brown] is complex symmetric. In this paper, we use the term only in the sense of Campbell.

As we know binormality and complex symmetry are meaningful for matrices. those properties automatically transfer to the adjoint of an operator. Furthermore, they have joint connections to square roots of normals operators. For those reasons, we investigate binormality and complex symmetry.

In this paper, first, we describe the conditions that a binormal operator becomes a normal operator and an n-binormal becomes an n-normal. we also give some properties of binormal, n-binormal and n-normal. Second, we solve the problem that binormal is not closed under addition. Third, we describe necessary and sufficient conditions that a skew complex symmetric operator becomes a binormal opetrator.

2 Some Properties of Binormal, n-binormal and n-normal

In this section, we give some properties of binormal, n-binormal and n-normal. We describe the conditions that a binormal operator becomes a normal operator and an n-binormal becomes an n-normal. We all know that normal operators and quasinormal operators are binormal operators, the converse is not true. we give the following theorem.

Theorem 1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a binormal operator. T satisfies with $T = CT^*C$, where C is a conjugation on $\mathcal{H}$. If C commutes with T^*T and TT^*, then T is a normal operator.*

Proof. Since T is a binormal and complex symmetric operator, we obtain $T = CT^*C$ and $T^*TTT^* = TT^*T^*T$, so

$$\begin{aligned} T^*TTT^* &= TT^*T^*T \\ T^*TCT^*CCTC &= TT^*CTCCT^*C \\ T^*TCT^*TC &= TT^*CTT^*C. \end{aligned}$$

Since, C commutes with T^*T and TT^*, we have $T^*TT^*T = TT^*TT^*$, therefore $TT^* = T^*T$. Hence, T is a normal operator. □

It is obvious that T is n-normal, then T is n-binormal, in the following theorem, we will give connections between n-binormal operators and n-normal operators.

Theorem 2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-binormal operator. T satisfies with $T = CT^*C$, where C is a conjugation on $\mathcal{H}$. If C commutes with T^*T^n, T^nT^* and T^{n-1}, then T is an n-normal operator.*

Proof. Since T is an n-binormal and complex symmetric operator, we obtain $T = CT^*C$ and $T^*T^nT^nT^* = T^nT^*T^*T^n$, then

$$\begin{aligned} T^*T^nT^nT^* &= T^nT^*T^*T^n \\ T^*T^nCT^{n*}CCTC &= T^nT^*CTCCT^{n*}C \\ T^*T^nCT^{n-1*}CTCTC &= T^nT^*CTCTCT^{n-1*}C \\ T^*T^nCT^{n-2*}CT^2CTC &= T^nT^*CTCT^2CT^{n-2*}C. \end{aligned}$$

So $T^*T^nCT^*CT^{n-1}CTC = T^nT^*CTCT^{n-1}CT^*C$.

Since C commutes with T^{n-1}, we have $T^*T^nCT^*CCT^nC = T^nT^*CT^nCCT^*C$. Since C commutes with T^*T^n and T^nT^*, we obtain $T^*T^nT^*T^nCC = T^nT^*T^nT^*CC$, therefore, $T^*T^n = T^nT^*$. Hence, T is an n-normal operator. □

Theorem 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (n-1)-normal operator. T satisfies with $T = CT^*C$, where C is a conjugation on $\mathcal{H}$. If C commutes with TT^*, then T is an n-normal operator.*

Proof. Since T is an (n-1)-normal and complex symmetric operator, we obtain $T = CT^*C$ and $T^*T^{n-1} = T^{n-1}T^*$, we have $T^*T^{n-1}T = T^{n-1}T^*T$. Then, $T^*T^n = T^{n-1}T^*T = T^{n-1}CTCCT^*C = T^{n-1}CTT^*C$. Since C commutes with TT^*, we have $T^*T^n = T^{n-1}TT^*CC = T^nT^*$. Hence, T is an n-normal operator. □

Theorem 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (n-1)-normal operator. T satisfies with $T = CT^*C$, where C is a conjugation on $\mathcal{H}$. If C commutes with TT^*, then T is an n-binormal operator.*

Proof. Since T is an (n-1)-normal and complex symmetric operator, we obtain $T = CT^*C$ and $T^*T^{n-1} = T^{n-1}T^*$. we have

$$\begin{aligned} T^*T^{n-1}T^{n-1}T^* &= T^{n-1}T^*T^*T^{n-1} \\ TT^*T^{n-1}T^{n-1}T^*T &= TT^{n-1}T^*T^*T^{n-1}T \\ CT^*CCTCT^{n-1}T^{n-1}CTCCT^*C &= T^nT^*T^*T^n \\ CT^*TCT^{n-1}T^{n-1}CTT^*C &= T^nT^*T^*T^n. \end{aligned}$$

Since C commutes with TT^*, we have $T^*T^nT^nT^* = T^nT^*T^*T^n$. Hence, T is an n-binormal operator. □

Theorem 5. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with conjugation C, If C commutes with T^n, then T is an n-binormal operator.*

Proof. Since $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric with conjugation C, we have $T = CT^*C$ and $CT^nC = T^{n*}$. Since C commutes with T^n. We can obtain

$$\begin{aligned} T^*T^nT^nT^* &= T^*T^nCCT^nT^* = T^*CT^{n*}CCCT^nT^* \\ &= T^*CT^{n*}CCCT^nCTC = T^{n+1*}CT^{n+1}C \\ &= CT^nTCCTT^nC = T^nCTCCTCT^n \\ &= T^nT^*T^*T^n. \end{aligned}$$

Hence, T is an n-binormal operator. □

Theorem 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator, then T^2 is binormal.*

Proof. Since, $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, we have $T^*T = TT^*$

$$\begin{aligned} T^*T^*TTTTT^*T^* &= T^*TT^*TTT^*TT^* \\ &= TT^*T^*TTT^*T^*T \\ &= TTT^*T^*T^*TT^*T \\ &= TTT^*T^*T^*T^*TT. \end{aligned}$$

Hence, T^2 is a binormal operator. □

Theorem 7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator, then T^n is binormal.*

Proof. Since, $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, we have $T^*T = TT^*$.

We have:

$$\begin{aligned} T^{n*}T^nT^nT^{n*} &= T^{n-1*}T^*TT^{n-1}T^{n-1}TT^*T^{n-1*} \\ &= T^{n-1*}TT^*TT^{n-2}T^{n-2}TT^*TT^{n-1*} \\ &= T^{n-1}T^{n-1*}T^*TTT^*T^{n-1*}T^{n-1} \\ &= T^{n-1}T^{n-2*}T^*TT^*T^*TT^*T^{n-2*}T^{n-1} \\ &= T^{n-1}T^{n-2*}TT^{2*}T^{2*}TT^{n-2*}T^{n-1} \\ &= T^{n-1}T^{n-3*}TT^{3*}T^{3*}TT^{n-3*}T^{n-1}. \end{aligned}$$

By induction, we have $T^{n*}T^nT^nT^{n*} = T^{n-1}TT^*T^{n-1*}T^{n-1*}T^*TT^{n-1} = T^nT^{n*}T^{n*}T^n$.
Hence, T^n is a binormal operator. □

3 Binormal is Closed under an Additional Condition

From [5], we know that a binormal operator T has some good properties: (i) αT is binormal, where α is complex number. (ii) T^* and T^{-1} is binormal, if it exists. However, the binormal is not closed under an additional condition, we give an example.

Example 8. *[5] Let* $T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *it is easy to verify that* T_1 *and* T_2 *are binormal. Let* $T = T_1 + T_2$, *we obtain* $T^*TTT^* = TT^*T^*T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \neq 0$. *Hence,* T *is not binormal.*

The following theorem will solve this problem.

Theorem 9. *Let* $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ *be binormal and complex symmetric with conjugation* C. T_1 *and* T_2 *satisfy with* $T_1T_2 = T_2T_1$. *If* C *commutes with* $T_1T_1^*T_2^*T_1, T_2T_1^*T_2^*T_1, T_2T_1^*T_2^*T_2, T_1T_1^*T_2^*T_2, T_1T_1^{*2}T_2, T_2T_1^{*2}T_1, T_2T_1^{*2}T_2, T_1T_2^{*2}T_1, T_1T_2^{*2}T_2, T_2T_2^{*2}T_1$, *then* $T_1 + T_2$ *is a binormal operator.*

Proof. Since T_1 and T_2 are complex symmetric and binormal, we have $CT_1C = T_1^*$ and $CT_2C = T_2^*$. We also have $T_1^*T_1T_1T_1^* = T_1T_1^*T_1^*T_1$ and $T_2^*T_2T_2T_2^* = T_2T_2^*T_2^*T_2$. Let $T = T_1 + T_2$, we consider T^*TTT^* and TT^*T^*T.

$$
\begin{aligned}
& T^*TTT^* \\
= \ & (T_1^* + T_2^*)(T_1 + T_2)(T_1 + T_2)(T_1^* + T_2^*) \\
= \ & T_1^*T_1^2T_1^* + T_1^*T_1^2T_2^* + T_1^*T_2^2T_1^* + T_1^*T_2^2T_2^* + T_2^*T_1^2T_1^* + T_2^*T_1^2T_2^* + T_2^*T_2^2T_1^* \\
+ \ & T_2^*T_2^2T_2^* + 2T_1^*T_2T_1T_1^* + 2T_1^*T_1T_2T_2^* + 2T_2^*T_1T_2T_1^* + 2T_2^*T_1T_2T_2^*.
\end{aligned}
$$

$$
\begin{aligned}
& TT^*T^*T \\
= \ & (T_1 + T_2)(T_1^* + T_2^*)(T_1^* + T_2^*)(T_1 + T_2) \\
= \ & T_1T_1^{*2}T_1 + T_1T_1^{*2}T_2 + T_1T_2^{*2}T_1 + T_1T_2^{*2}T_2 + T_2T_1^{*2}T_1 + T_2T_1^{*2}T_2 + T_2T_2^{*2}T_1 \\
+ \ & T_2T_2^{*2}T_2 + 2T_1T_1^*T_2^*T_1 + 2T_1T_1^*T_2^*T_2 + 2T_2T_1^*T_2^*T_1 + 2T_2T_1^*T_2^*T_2.
\end{aligned}
$$

Since $T_1^*T_1^2T_1^* = T_1T_1^{*2}T_1$, $T_2^*T_2^2T_2^* = T_2T_2^{*2}T_2$, T_1 and T_2 satisfy with $T_1T_2 = T_2T_1$.

Since C commutes with $T_1T_1^*T_2^*T_1$, we have:

$$
\begin{aligned}
T_1^*T_1T_2T_1^* &= CT_1CCT_1^*CCT_2^*CCT_1C = CT_1T_1^*T_2^*T_1C \\
&= T_1T_1^*T_2^*T_1CC = T_1T_1^*T_2^*T_1.
\end{aligned}
$$

Since C commutes with $T_1T_1^*T_2^*T_2$, we have:

$$
\begin{aligned}
T_1^*T_1T_2T_2^* &= CT_1CCT_1^*CCT_2^*CCT_2C = CT_1T_1^*T_2^*T_2C \\
&= T_1T_1^*T_2^*T_2CC = T_1T_1^*T_2^*T_2.
\end{aligned}
$$

Since C commutes with $T_2T_1^*T_2^*T_1$, we have:

$$\begin{aligned} T_2^*T_1T_2T_1^* &= CT_2CCT_1^*CCT_2^*CCT_1C = CT_2T_1^*T_2^*T_1C \\ &= T_2T_1^*T_2^*T_1CC = T_2T_1^*T_2^*T_1. \end{aligned}$$

Since C commutes with $T_2T_1^*T_2^*T_2$, we have:

$$\begin{aligned} T_2^*T_1T_2T_2^* &= CT_2CCT_1^*CCT_2^*CCT_2C = CT_2T_1^*T_2^*T_2C \\ &= T_2T_1^*T_2^*T_2CC = T_2T_1^*T_2^*T_2. \end{aligned}$$

Since C commutes with $T_1T_1^{*2}T_2$, we have:

$$\begin{aligned} T_1^*T_1^2T_2^* &= CT_1CCT_1^{*2}CCT_2C = CT_1T_1^{*2}T_2C \\ &= T_1T_1^{*2}T_2CC = T_1T_1^{*2}T_2. \end{aligned}$$

Since C commutes with $T_1T_2^{*2}T_1$, we have:

$$\begin{aligned} T_1^*T_2^2T_1^* &= CT_1CCT_2^{*2}CCT_1C = CT_1T_2^{*2}T_1C \\ &= T_1T_2^{*2}T_1CC = T_1T_2^{*2}T_1. \end{aligned}$$

Since C commutes with $T_1T_2^{*2}T_2$, we have:

$$\begin{aligned} T_1^*T_2^2T_2^* &= CT_1CCT_2^{*2}CCT_2C = CT_1T_2^{*2}T_2C \\ &= T_1T_2^{*2}T_2CC = T_1T_2^{*2}T_2. \end{aligned}$$

Since C commutes with $T_2T_1^{*2}T_1$, we have:

$$\begin{aligned} T_2^*T_1^2T_1^* &= CT_2CCT_1^{*2}CCT_1C = CT_2T_1^{*2}T_1C \\ &= T_2T_1^{*2}T_1CC = T_2T_1^{*2}T_1. \end{aligned}$$

Since C commutes with $T_2T_1^{*2}T_2$, we have:

$$\begin{aligned} T_2^*T_1^2T_2^* &= CT_2CCT_1^{*2}CCT_2C = CT_2T_1^{*2}T_2C \\ &= T_2T_1^{*2}T_2CC = T_2T_1^{*2}T_2. \end{aligned}$$

Since C commutes with $T_2T_2^{*2}T_1$, we have:

$$\begin{aligned} T_2^*T_2^2T_1^* &= CT_2CCT_2^{*2}CCT_1C = CT_2T_2^{*2}T_1C \\ &= T_2T_2^{*2}T_1CC = T_2T_2^{*2}T_1. \end{aligned}$$

Therefore, we obtain $T^*TTT^* = TT^*T^*T$. Hence, $T_1 + T_2$ is a binormal operator. □

4 Conditions that Skew Complex Symmetric Operators become Binormal or Quasinormal

From [8], we can learn about some results of complex symmetric, binormal operators. We will give some results of skew complex symmetric, binormal operators.

Theorem 10. *Let $T \in \mathcal{L}(\mathcal{H})$ be a skew complex symmetric operator with conjugation C. T is a binormal operator, if and only if C commutes with TT^*T^*T.*

Proof. Since T is skew complex symmetric, then $CTC = -T^*$. We assume that C commutes with TT^*T^*T, then

$$\begin{aligned} T^*TTT^* &= T^*TCCTT^* = T^* - CT^* - T^*CT^* \\ &= CTT^*T^* - CT^* = CTT^*T^*TC \\ &= TT^*T^*TCC = TT^*T^*T. \end{aligned}$$

Conversely, suppose T is binormal, then

$$\begin{aligned} T^*TTT^*C &= T^*TCCTT^*C = T^* - CT^* - T^*CT^*C \\ &= CTT^*T^* - CT^*C = CTT^*T^*T. \end{aligned}$$

□

Theorem 11. *Let $T \in \mathcal{L}(\mathcal{H})$ be a skew complex symmetric operator with conjugation C, if C commutes with T^*T. Then T is quasinormal.*

Proof. Since T is skew complex symmetric, then $CTC = -T^*$.

$$\begin{aligned} TT^*T &= TCCT^*T = -CT^* - TCT \\ &= CT^*TCT = CCT^*TT \\ &= T^*TT. \end{aligned}$$

Hence, T is quasinormal. □

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