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Some Inequalities for the p-Digamma Function

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Abstract

Some inequalities involving the p-digamma function are presented. These results are the p-analogues of some recent results.

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1 Introduction and Preliminaries

The classical Euler's Gamma function $\Gamma(t)$ and the digamma function $\psi(t)$ are commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \quad \text{and} \quad \psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

Similarly the p-Gamma and p-digamma functions are defined as (see [1])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0.$$

and

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \qquad t > 0.$$

The functions $\psi(t)$ and $\psi_p(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t > 0.$$
$$\psi_p(t) = \ln p - \sum_{n=0}^{p}\frac{1}{n+t}, \quad p \in N, \quad t > 0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of the above functions, we arrive at the following statements for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t > 0.$$
$$\psi_p^{(m)}(t) = (-1)^{m-1}m!\sum_{n=0}^{p}\frac{1}{(n+t)^{m+1}}, \quad p \in N, \quad t > 0.$$

In 2011, Sulaiman [3] presented the following results.

$$\psi(t+s) \geq \psi(t) + \psi(s) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(t+s) \leq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{2}$$

where m is a positive odd integer and $t, s > 0$.

$$\psi^{(m)}(t+s) \geq \psi^{(m)}(t) + \psi^{(m)}(s) \tag{3}$$

where m is a positive even integer and $t, s > 0$.

In a recent paper, Sroysang [2] presented the following geralizations of the above inequalities.

$$\psi(t + \sum_{i=1}^{\alpha}\beta_i s_i) \geq \psi(t) + \sum_{i=1}^{\alpha}\beta_i\psi(s_i) \tag{4}$$

where $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha}\beta_i s_i) \leq \psi^{(m)}(t) + \sum_{i=1}^{\alpha}\beta_i\psi^{(m)}(s_i) \tag{5}$$

where m is a positive odd integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

$$\psi^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi^{(m)}(s_i) \tag{6}$$

where m is a positive even integer, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$.

The objective of this paper is to establish that the inequalities (4), (5) and (6) still hold true for the p-digamma function.

2 Main Results

We now present our results.

Theorem 2.1. *Let $p \in N$, $t > 0$, $\beta_i > 0$ and $0 < s_i < 1$ for all $i \in N_\alpha$. Then the following inequality is valid.*

$$\psi_p(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_p(t) + \sum_{i=1}^{\alpha} \beta_i \psi_p(s_i) \tag{7}$$

Proof. Let $f(t) = \psi_p(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} f'(t) &= \psi_p'(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p'(t) \\ &= \sum_{n=0}^{p} \left[\frac{1}{(n + t + \sum_{i=1}^{\alpha} \beta_i s_i)^2} - \frac{1}{(n+t)^2} \right] \leq 0 \end{aligned}$$

That implies f is non-increasing. In addition,

$$\begin{aligned} \lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \left[\psi_p(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p(s_i) \right] \\ &= -\ln p \sum_{i=1}^{\alpha} \beta_i \\ &+ \lim_{t\to\infty} \left[-\sum_{n=0}^{p} \frac{1}{(n + t + \sum_{i=1}^{\alpha} \beta_i s_i)} + \sum_{n=0}^{p} \frac{1}{(n+t)} + \sum_{n=0}^{p} \sum_{i=1}^{\alpha} \frac{\beta_i}{(n + s_i)} \right] \\ &= -\ln p \sum_{i=1}^{\alpha} \beta_i + \sum_{n=0}^{p} \sum_{i=1}^{\alpha} \frac{\beta_i}{(n + s_i)} \geq 0. \end{aligned}$$

Therefore $f(t) \geq 0$ yielding the result.

Theorem 2.2. *Let $p \in N$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_p^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \leq \psi_p^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i) \tag{8}$$

Proof. Let $g(t) = \psi_p^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i)$. Then fixing s_i for each i we have,

$$\begin{aligned} g'(t) &= \psi_p^{(m+1)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p^{(m+1)}(t) \\ &= (-1)^m (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right] \\ &= -(m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right], \text{ (for odd } m) \\ &\geq 0. \end{aligned}$$

That implies g is non-decreasing. In addition,

$$\begin{aligned} \lim_{t\to\infty} g(t) &= \lim_{t\to\infty} \left[\psi_p^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i) \right] \\ &= (-1)^{m-1} m! \times \\ &\lim_{t\to\infty} \sum_{n=0}^{p} \left[\frac{1}{(n + t + \sum_{i=1}^{\alpha} \beta_i s_i)^{m+1}} - \frac{1}{(n+t)^{m+1}} - \sum_{i=1}^{\alpha} \beta_i \frac{1}{(n+s_i)^{m+1}} \right] \\ &= -m! \sum_{n=0}^{p} \sum_{i=1}^{\alpha} \frac{\beta_i}{(n+s_i)^{m+1}} \leq 0. \quad \text{(since } m \text{ is odd)} \end{aligned}$$

Therefore $g(t) \leq 0$ yielding the result.

Theorem 2.3. *Let $p \in N$, $t > 0$, $\beta_i > 0$ and $s_i > 0$ for all $i \in N_\alpha$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_p^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) \geq \psi_p^{(m)}(t) + \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i) \tag{9}$$

Proof. Let $h(t) = \psi_p^{(m)}(t + \sum_{i=1}^{\alpha} \beta_i s_i) - \psi_p^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i)$. Then fixing

s_i for each i we have,

$$\begin{aligned}
h'(t) &= \psi_p^{(m+1)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_p^{(m+1)}(t) \\
&= (-1)^m (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right] \\
&= (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right], \text{ (for even } m) \\
&\leq 0.
\end{aligned}$$

That implies h is non-increasing. In addition,

$$\begin{aligned}
\lim_{t\to\infty} g(t) &= \lim_{t\to\infty} \left[\psi_p^{(m)}(t+\sum_{i=1}^{\alpha}\beta_i s_i) - \psi_p^{(m)}(t) - \sum_{i=1}^{\alpha} \beta_i \psi_p^{(m)}(s_i) \right] \\
&= (-1)^{m-1} m! \times \\
&\lim_{t\to\infty} \sum_{n=0}^{p} \left[\frac{1}{(n+t+\sum_{i=1}^{\alpha}\beta_i s_i)^{m+1}} - \frac{1}{(n+t)^{m+1}} - \sum_{i=1}^{\alpha} \beta_i \frac{1}{(n+s_i)^{m+1}} \right] \\
&= m! \sum_{n=0}^{p} \sum_{i=1}^{\alpha} \frac{\beta_i}{(n+s_i)^{m+1}} \geq 0. \quad \text{(since } m \text{ is even)}
\end{aligned}$$

Therefore $h(t) \geq 0$ yielding the result.

Remark 2.4. If we let $p \to \infty$ in inequalities (7), (8) and (9) then we repectively recover the inequalities (4), (5) and (6).

References

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