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Some inequalities for the n-th Derivative of the Incomplete Zeta Functions

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Abstract

Recently, we gave new inequalities involving the n-th derivative of the incomplete zeta functions. In this paper, we present some inequalities for the n-th derivative of the incomplete zeta functions.

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1 Introduction

In [2], the incomplete zeta function $\xi_{a,b}$ is defined by

$$\xi_{a,b}(s) = \frac{1}{\Gamma(s)} \int_a^b \frac{t^{s-1}}{e^t - 1} dt,$$

for any $0 \leq a < b$ and $s > 1$, where Γ is the gamma function.

In [1], for any $0 \leq a < b$, we define the function $h_{a,b}(x)$ by

$$h_{a,b}(x) = \int_a^b \frac{t^{x-1}}{e^t - 1} dt$$

for all $x > 1$.

It is easy to see on the n-th derivative of $h_{a,b}$ that, for any $0 \leq a < b$,

$$h_{a,b}^{(n)}(x) = \int_a^b \frac{(\log_e t)^n t^{x-1}}{e^t - 1} dt$$

for all $x > 1$.

Moreover, we [1] gave new inequalities involving the n-th derivative of the incomplete zeta function as follows.

Theorem 1.1. [1] *Let $0 \leq a < b$ and $y > 0$ and let n be a positive odd integer. Then*
$$h_{a,b}^{(n)}(x+y) \geq h_{a,b}^{(n)}(x)$$
for all $x > 1$.

Theorem 1.2. [1] *Let $0 \leq a < b$ and $x_1, x_2, ..., x_n > 1$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*
$$\left(h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x_i).$$

Theorem 1.3. [1] *Let $1 < a < b$ and $x_1, x_2, ..., x_n > 1$ and let $k_1, k_2, ..., k_n$ be non-negative integers and let $k = \sum_{i=1}^{n} k_i$. Then*
$$\left(h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(x_i).$$

In this paper, we present some inequalities for the n-th derivative of the incomplete zeta functions.

2 Results

Theorem 2.1. *Let $0 \leq a < b$ and $y > 0$ and let n be a positive odd integer. Then*
$$h_{a,b}^{(n)}(x-y) \leq h_{a,b}^{(n)}(x) \tag{1}$$
for all $x > 1 + y$.

Proof. For any $x > 1 + y$,
$$\begin{aligned} h_{a,b}^{(n)}(x) - h_{a,b}^{(n)}(x-y) &= \int_a^b \frac{(\log_e t)^n t^{x-1}}{e^t - 1} dt - \int_a^b \frac{(\log_e t)^n t^{x-y-1}}{e^t - 1} dt \\ &= \int_a^b \frac{(\log_e t)^n}{e^t - 1} \left(t^{x-1} - t^{x-y-1}\right) dt \\ &= \int_a^b \frac{(\log_e t)^{n-1}}{e^t - 1} (\log_e t) \left(t^{x-1} - t^{x-y-1}\right) dt \\ &\geq 0. \end{aligned}$$

This implies the inequality (1). □

Theorem 2.2. *Let $0 \leq a < b$ and $x_1, x_2, ..., x_n > 0$ and let $k_1, k_2, ..., k_n$ be non-negative even integers and let $k = \sum_{i=1}^{n} k_i$. Then*

$$\left(h_{a,b}^{(k)}\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(1+x_i). \tag{2}$$

Proof. By the assumption,

$$\begin{aligned}
h_{a,b}^{(k)}\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right) &= \int_a^b \frac{(\log_e t)^k t^{\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right)-1}}{e^t-1}dt \\
&= \int_a^b \frac{(\log_e t)^k t^{\sum_{i=1}^{n}\frac{x_i}{n}}}{e^t-1}dt \\
&= \int_a^b \frac{(\log_e t)^{\sum_{i=1}^{n}k_i} t^{\sum_{i=1}^{n}\frac{x_i}{n}}}{\left((e^t-1)^{1/n}\right)^n}dt \\
&= \int_a^b \frac{\prod_{i=1}^{n}(\log_e t)^{k_i}\prod_{i=1}^{n} t^{\frac{x_i}{n}}}{\left((e^t-1)^{1/n}\right)^n}dt \\
&= \int_a^b \prod_{i=1}^{n}\frac{(\log_e t)^{k_i} t^{\frac{x_i}{n}}}{(e^t-1)^{1/n}}dt \\
&= \int_a^b \prod_{i=1}^{n}\left(\frac{(\log_e t)^{nk_i} t^{x_i}}{e^t-1}\right)^{1/n}dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned}
h_{a,b}^{(k)}\left(1+\sum_{i=1}^{n}\frac{x_i}{n}\right) &\leq \prod_{i=1}^{n}\left(\int_a^b \frac{(\log_e t)^{nk_i} t^{x_i}}{e^t-1}dt\right)^{1/n} \\
&= \prod_{i=1}^{n}\left(\int_a^b \frac{(\log_e t)^{nk_i} t^{1+x_i-1}}{e^t-1}dt\right)^{1/n} \\
&= \prod_{i=1}^{n}\left(h^{(nk_i)}(1+x_i)\right)^{1/n} \\
&= \left(\prod_{i=1}^{n} h^{(nk_i)}(1+x_i)\right)^{1/n}.
\end{aligned}$$

This implies the inequality (2). ∎

Corollary 2.3. *Let* $0 \leq a < b$ *and* $x > 0$ *and let* $k_1, k_2, ..., k_n$ *be non-negative even integers and let* $k = \sum_{i=1}^{n} k_i$. *Then*

$$\left(h_{a,b}^{(k)}(1+x)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(1+x).$$

Proof. This follows from Theorem 2.2 in case $x_1 = x_2 = ... = x_n$ □

Theorem 2.4. *Let* $1 < a < b$ *and* $x_1, x_2, ..., x_n > 0$ *and let* $k_1, k_2, ..., k_n$ *be non-negative integers and let* $k = \sum_{i=1}^{n} k_i$. *Then*

$$\left(h_{a,b}^{(k)}\left(\sum_{i=1}^{n} \frac{1+x_i}{n}\right)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(1+x_i). \tag{3}$$

Proof. This proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let* $1 < a < b$ *and* $x > 0$ *and let* $k_1, k_2, ..., k_n$ *be non-negative integers and let* $k = \sum_{i=1}^{n} k_i$. *Then*

$$\left(h_{a,b}^{(k)}(1+x)\right)^n \leq \prod_{i=1}^{n} h_{a,b}^{(nk_i)}(1+x).$$

Proof. This follows from Theorem 2.4 in case $x_1 = x_2 = ... = x_n$ □

References

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