

# **Some Fixed Point Results on Cone Metric Spaces with w-Distance**

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## **Abstract**

The aim of this paper is to consider cone metric spaces with w-distance and obtained fixed point theorems on cone metric space with w-distance .Our results generalize and unify some well known results.

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**Keywords:** Cone metric space with w-distance, weakly contractive type mappings.

## 1 Introduction

In 2007 Huang and Zhang [1] announced the notion of cone metric spaces and obtained some fixed point theorems for different type of contractive mappings using the normality of cone. On the other hand Kada et al [2] and Shioji et al [3] introduced the concept of metric space with  $w$ -distance, gave some examples, properties of  $w$ -distance and improved Caristi's fixed point [6], Ekeland's variational principle [7] and the non-convex minimization theorem according to Takahashi [8]. By the use of the concept of  $w$ -distance also proved a fixed point theorem in complete metric space which generalized the fixed point theorems of Subrahmanyam [9], Kanan [10] and Ćirić [11]. Recently Lakijan and Arabyani [4] combined these concepts together and introduced the concept of cone metric spaces with  $w$ -distances and established some fixed point theorems. In the sequel Dhanorkar and Salunke [5] proved some fixed point theorems on cone metric space with  $w$ -distance.

In this paper we also obtained fixed point theorems on cone metric space with  $w$ -distance. Our results generalize and extend the result of [4], [5].

## 2 General Framework

**Definition 2.1** Let  $E$  always be real Banach space and  $P$  a subset of  $E$ .  $P$  is called a cone if

- (i)  $P$  is closed, non-empty and  $P \neq 0$ .
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and non-negative real numbers  $a, b$ .
- (iii)  $P \cap (-P) = 0$ .

For a given cone  $P \subseteq E$ , we can define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ ,  $x < y$  will stand for  $x \leq y$  and  $x \neq y$ , while  $x < y$  will stand for  $y - x \in \text{int } P$ , where  $\text{int } P$  denotes the interior of  $P$ .

**Definition 2.2** The cone  $P$  is called normal if there is a number  $M > 0$  such that for all  $x, y \in E$ ,  $0 \leq x \leq y$  implies

$$\|x\| \leq M \|y\|.$$

The least positive number satisfying above is called the normal constant of  $P$  ([1]).

**Definition 2.3** The cone  $P$  is called regular if every increasing sequence which is bounded from above is convergent. That is, if  $\{x_n\}_{n \geq 1}$  is a sequence such that  $x_1 \leq x_2 \leq \dots \leq y$  for some  $y \in E$ , then there is  $x \in E$  such that  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ .

Equivalently the cone  $P$  is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been mentioned that every regular cone is normal.

**Lemma 2.4** Every regular cone is normal.

In the following we always suppose that  $E$  is a Banach space,  $P$  is a cone in  $E$  with  $\text{int } P \neq 0$  and  $\leq$  is partial ordering with respect to  $P$ .

**Definition 2.5** Let  $X$  be a non-empty set. Suppose the mapping  $d : X \times X \rightarrow E$  satisfies :

- (a)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (b)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (c)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ . Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space.

**Example 2.6** Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|, \alpha|x - y|)$ , where  $\alpha \geq 0$  is constant. Then  $(X, d)$  is a cone metric space [1].

**Definition 2.7** Let  $X$  be a cone metric space with metric  $d$ . Then a mapping  $p : X \times X \rightarrow E$  is called  $w$ -distance on  $X$  if the following satisfy :

- (a)  $0 \leq p(x, y)$  for all  $x, y \in X$ .
- (b)  $p(x, z) \leq p(x, y) + p(y, z)$ , for all  $x, y, z \in X$ .
- (c)  $p(x, z) \rightarrow E$  is lower semi-continuous for all  $x \in X$ .
- (d) for any  $0 < \alpha$ , there exists  $0 < \beta$  such that  $p(z, x) < \beta$  and  $p(z, y) < \beta$  imply  $d(x, y) < \alpha$ , where  $\alpha, \beta \in E$ .

**Lemma 2.8** Let  $X$  be a cone metric space with metric  $d$ , let  $p$  be a  $w$ -distance on  $X$  and let  $f$  be a function from  $X$  into  $E$  that  $0 \leq f(x)$  for any  $x \in X$ . Then a into function  $q : X \times X \rightarrow E$  given by  $q(x, y) = f(x) + p(x, y)$  for each  $(x, y) \in X \times X$  is also a  $w$ -distance.

**Example 2.9** (i) Let  $(X, d)$  be a metric space. Then  $p = d$  is a  $w$ -distance on  $X$ .

- (ii) Let  $X$  be norm linear space with Euclidean norm. Then the mapping  $p : X \times X \rightarrow [0, \infty)$  defined by  $p(x, y) = \|x\| + \|y\|$ , for all  $x, y \in X$  is a  $w$ -distance on  $X$ .

- (iii) Let  $X$  be norm linear space with Euclidean norm. Then the mapping  $p: X \times X \rightarrow [0, \infty)$  defined by  $p(x, y) = \|y\|$ , for all  $x, y \in X$  is a  $w$ -distance on  $X$ .

**Definition 2.10** Let  $X$  be a cone metric space with metric  $d$ , let  $p$  be a  $w$ -distance on  $X$ ,  $x \in X$  and  $\{x_n\}$  a sequence in  $X$ , then

- (a)  $\{x_n\}$  is called a  $p$ -Cauchy sequence whenever for every  $\alpha \in E$ ,  $0 < \alpha$ , there is a positive integer  $N$  such that, for all  $m, n \geq N$ ,  $p(x_m, x_n) < \alpha$ .
- (b) A sequence  $\{x_n\}$  in  $X$  is called a  $p$ -convergent to a point  $x \in X$  whenever for every  $\alpha \in E$ ,  $0 < \alpha$ , there is a positive integer  $N$  such that, for all  $n \geq N$ ,  $p(x, x_n) < \alpha$ . Note that by lower semi-continuous  $p$ , for all  $n \geq N$ ,  $p(x_n, x) < \alpha$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$ .
- (c)  $(X, d)$  is a complete cone metric space with  $w$ -distance if every Cauchy sequence is  $p$ -convergent.

Finally, note that the relations  $\text{int } P + \text{int } P \in \text{int } P$  and  $\lambda \text{int } P \subseteq \text{int } P$  ( $X > 0$ ).

**Lemma 2.11** There is not normal cone with normal constant  $M < 1$ .

**Example 2.12** Let  $E = I^1$ ,  $P = \{\{x_n\}_{n \geq 1} \in E : x_n \geq 0, \text{ for all } n\}$ ,  $(X, p)$  a metric space and  $d: X \times X \rightarrow E$  defined by  $d(x, y) = \left\{ \frac{p(x, y)}{2^n} \right\}_{n=1}$ . Then  $(X, d)$  is a cone metric space if set  $p = d$  then  $(X, d)$  is cone metric space with  $w$ -distance  $p$  and the normal constant of  $P$  is equal to  $M = 1$ . For each  $k > 1$ , there is a normal cone with normal constant  $M > k$ .

### 3 Main Results

**Theorem 3.1** Let  $(X, d)$  be a complete cone metric space with  $w$ -distance  $p$ ,  $P$  be a normal cone on  $X$ . Suppose a mapping  $T: X \rightarrow X$  satisfy the contractive condition.

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta [p(x, Tx) + p(y, Ty)] + \gamma [p(x, Ty) + p(y, Tx)] \quad (3.1)$$

for all  $x, y \in X$ .  $\alpha, \beta, \gamma$  are non negative reals such that  $\alpha + 2\beta + 2\gamma < 1$ . Then  $T$  has a unique fixed point in  $X$ . For each  $x \in X$  the iterative sequence  $\{T^n(x)_{n \geq 1}\}$  converges the fixed point.

**Proof:** Let  $x_0 \in X$  be arbitrary. Define a sequence  $\{x_n\}$  in  $X$  such that

$$x_1 = Tx_0, \quad x_2 = Tx_1 = T^2x_0, \dots, x_n = T^n x_0.$$

By condition 3.1, we have,

$$\begin{aligned} p(Tx_0, T^2x_0) &= p[Tx_0, T(Tx_0)] \leq \alpha p(x_0, Tx_0) \\ &\quad + \beta [p(x_0, Tx_0) + p(Tx_0, T^2x_0)] \\ &\quad + \gamma [p(x_0, T^2x_0) + p(Tx_0, Tx_0)] \end{aligned}$$

this implies

$$\begin{aligned} &\leq (\alpha + \beta + \gamma)p(x_0, Tx_0) + (\beta + \gamma)[p(Tx_0, T^2x_0)] \\ p(Tx_0, T^2x_0) &\leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} p(x_0, Tx_0) \\ &\leq Kp(x_0, Tx_0) \end{aligned}$$

where  $K = \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} < 1$

again

$$p(T^2x_0, T^3x_0) \leq Kp(Tx_0, T^2x_0) \leq K^2p(x_0, Tx_0) \leq \dots$$

continue in this way, we get

$$p(T^n x_0, T^{n+1} x_0) \leq K^n p(x_0, Tx_0)$$

where

$$0 \leq K < 1$$

Let  $m, n \in \mathbb{N}$  and  $m > n$ , we have,

$$\begin{aligned} p(T^n x_0, T^m x_0) &\leq [p(T^n x_0, T^{n+1} x_0) + p(T^{n+1} x_0, T^{n+2} x_0) + \dots + p(T^{m-1} x_0, T^m x_0)] \\ &\leq (K^n + K^{n+1} + \dots + K^{m-1})p(x_0, Tx_0) \\ &\leq \frac{K^n}{1 - K} p(x_0, Tx_0) \end{aligned}$$

Since  $0 \leq K < 1$  and  $n \rightarrow \infty$ , then  $K^n \rightarrow 0$ . Therefore  $\{T^n x_0\}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete cone metric space then there exists  $u \in X$ , such that  $x_n \rightarrow u$  as  $n \rightarrow \infty$ .

Hence

$$\begin{aligned} p(Tu, u) &\leq p(Tu, Tx_n) + p(Tx_n, u) \\ &\leq \alpha p(u, x_n) \\ &\quad + \beta [p(u, Tu) + p(x_n, Tx_n)] \\ &\quad + \gamma [p(u, Tx_n) + p(Tx_n, Tu)] + p(Tx_n, u). \end{aligned}$$

$$p(Tu, u) \leq \left\{ \frac{\alpha}{1 - \beta} p(u, x_n) + \frac{\beta}{1 - \beta} p(x_n, Tx_n) \right\}$$

$$+ \frac{\gamma}{1-\beta} p(Tx_n, Tu) + \frac{\gamma+1}{1-\beta} p(Tx_n, u) \} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore  $Tu = u$ . Hence  $u$  is a fixed point of  $T$ .

**Uniqueness:** Let  $v$  be another fixed point of  $T$  i.e.  $Tv = v$ .

$$\begin{aligned} p(Tu, Tv) &= p(u, v) \\ &\leq \alpha p(u, v) + \beta [p(u, Tu) + p(v, Tv)] + \gamma [p(u, Tv) + p(v, Tu)] \\ &\leq \alpha p(u, v) + \beta [p(u, u) + p(v, v)] + \gamma [p(u, v) + p(v, u)] \\ p(u, v) &\leq (\alpha + 2\gamma) p(u, v) \end{aligned}$$

since  $\alpha + 2\gamma < 1$

Therefore

$$p(u, v) = 0 \quad \Rightarrow u = v$$

This shows that fixed point is unique.

This completes the proof.

#### 4 Remarks

1. In theorem 3.1 if we put  $\beta = \gamma = 0$  we get theorem 3.6 of [4].
2. In theorem 3.1 if we put  $\alpha = \gamma = 0$ , we get theorem 3.2 of [5].

If we put  $\alpha = \beta = 0$  we get the following result –

**Corollary 4.1 :** Let  $(X, d)$  be a complete cone metric space with  $w$ -distance.  $P$  be a normal cone on  $X$ . suppose a mapping  $T : X \rightarrow X$  satisfy the condition-

$$p(Tx, Ty) \leq \gamma [p(x, Ty) + p(y, Tx)]$$

for all  $x, y \in X$  and  $\gamma \in [0, \frac{1}{2})$  is a constant. Then  $T$  has a unique fixed point  $X$ , for each  $x \in X$ , the iterative sequence  $\{T_n(x)\}_{n \geq 1}$  converges to the fixed point.

If we put  $p=d$  in theorem 3.1 we obtain the following result

**Corollary 4.2 :** Let  $(X, d)$  be a complete cone metric space,  $P$  be a normal cone on  $X$ . Suppose a mapping  $T : X \rightarrow X$  satisfy the contractive condition.

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta [d(x, Tx) + d(y, Ty)] + \gamma [d(x, Ty) + d(x, Tx)]$$

for all  $x, y \in X$ .  $\alpha, \beta, \gamma$  are non negative reals such that  $\alpha + 2\beta + 2\gamma < 1$ . Then  $T$  has a unique fixed point in  $X$ . for each  $x \in X$  the iterative sequence  $\{T^n(x)_{n \geq 1}\}$  converges the fixed point.

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