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Some characterizations of strongly convex stochastic processes

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Abstract

In this paper some well-known characterizations of convex functions are extended to convex and strongly convex stochastic processes.

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1 Introduction

In 1980 K. Nikodem [5] introduced and investigated properties of convex stochastic processes. Next, A. Skowronski described the properties of Jensen-convex stochastic processes in [8]. Later, the Hermite-Hadamard inequality for convex stochastic processes was proved in [1].

In this article we will present the counterpart for convex and strongly convex stochastic processes of the well-known theorems from real analysis, which characterize the convex and strongly convex functions.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process, such that $E[X(t)]^2 < \infty$ for all $t \in I$, where $E[X(t)]$ denotes the expectation value of $X(t, \cdot)$. Recall that the stochastic process X is called

(i) *mean-square differentiable* in the interval I, if there exists a stochastic process X' (the derivative of X) such that for all $t_0 \in I$ we have

$$\lim_{t \to t_0} E\Big[\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\Big]^2 = 0;$$

(ii) *twice mean-square differentiable* in the interval I, if there exists a stochastic process X'' (the second derivative of X) such that for all $t_0 \in I$

$$\lim_{t\to t_0} E\Big[\frac{X'(t)-X'(t_0)}{t-t_0} - X''(t_0)\Big]^2 = 0;$$

(iii) *mean-square integrable* in $[a,b] \subset I$, if there exists a random variable Y such that for all normal sequence of partitions of the interval $[a,b]$ $a = t_0 < t_1 < t_2 < ... < t_n = b$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$, we have

$$\lim_{n\to\infty} E\Big[\sum_{k=1}^{n} X(\Theta_k)\cdot(t_k - t_{k-1}) - Y\Big]^2 = 0.$$

The random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a,b]$. We can also write

$$Y(\cdot) = \int_a^b X(s,\cdot)ds \quad \text{(a.e.)}.$$

For the definition and more basic properties of mean-square derivative and mean-square integral see [9].

Now, let $C : \Omega \to \mathbb{R}$ be a positive random variable. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be *strongly convex with modulus* $C(\cdot)$ if, for all $u, v \in I$ and for all $\lambda \in [0,1]$ the following inequality holds

$$X\big(\lambda u+(1-\lambda)v,\cdot\big) \leqslant \lambda X(u,\cdot)+(1-\lambda)X(v,\cdot)-C(\cdot)\lambda(1-\lambda)(u-v)^2 \quad \text{(a.e.)}. \tag{1}$$

For more details we refer to [2]. If we omit the term $C(\cdot)\lambda(1-\lambda)(u-v)^2$ in the inequality (1), we immediately get the definition of a convex stochastic process introduced by K. Nikodem in 1980 (see [5]).

The definition of strongly convex stochastic processes is motivated by the definition of strongly convex functions. Such functions play an important role in optimization theory and mathematical economies (see, for instance [6], [4], and the references therein).

In this paper we present some characterizations of strongly convex stochastic processes. These results generalize the ones for convex functions defined on real intervals involving the notions of support functions and the first and the second derivatives (cf. [7]; see also [3]).

2 Main results

We start our investigation with two very useful lemmas.

Lemma 2.1 ([2]). A stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly convex with modulus $C(\cdot)$ if and only if the stochastic process $Y : I \times \Omega \to \mathbb{R}$ defined by $Y(t, \cdot) := X(t, \cdot) - C(\cdot)t^2$ is convex.

Lemma 2.2. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if and only if X is supported at any point $t_0 \in intI$ by the process of the form $A(\cdot)(t - t_0) + X(t_0, \cdot)$, where $A : \Omega \to \mathbb{R}$ is a random variable.

Proof. Suppose that a process X is convex. By Proposition 2 from [1] we get the support of the form $A(\cdot)(t - t_0) + X(t_0, \cdot)$ at any point $t_0 \in intI$.
Let the process X has a support at any point $t_0 \in I$. It means the following inequality holds

$$X(t, \cdot) \geqslant A(\cdot)(t - t_0) + X(t_0, \cdot) \quad \text{(a.e.)}. \tag{2}$$

We fix $u, v \in I$ and $\lambda \in [0, 1]$, such that $t_0 = \lambda u + (1 - \lambda)v$. For u and v separately, by the inequality (2) we get

$$\lambda X(u, \cdot) \geqslant \lambda A(\cdot)(u - t_0) + \lambda X(t_0, \cdot) \quad \text{(a.e.)},$$

$$(1 - \lambda)X(v, \cdot) \geqslant (1 - \lambda)A(\cdot)(v - t_0) + (1 - \lambda)X(t_0, \cdot) \quad \text{(a.e.)}.$$

Adding by sides the above inequalities, we have

$$\lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \geqslant X(t_0, \cdot) \quad \text{(a.e.)}.$$

Finally, replacing t_0 by $\lambda u + (1 - \lambda)v$ we can write

$$\lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \geqslant X(\lambda u + (1 - \lambda)v, \cdot) \quad \text{(a.e.)}.$$

It completes the proof. □

Using the above lemmas, it can be easily shown the following theorem.

Theorem 2.3. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process. X is strongly convex with modulus $C(\cdot)$ if and only if for every $t_0 \in intI$ there exists the support of the form

$$C(\cdot)(t - t_0)^2 + A(\cdot)(t - t_0) + X(t_0, \cdot),$$

where $A : \Omega \to \mathbb{R}$ is a random variable.

Now, we will present a characterization of strongly convex processes via their first derivatives.

Lemma 2.4. Let $X : I \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process. X is convex in the interval I, if and only if the first derivative is nondecreasing in I.

Proof. Obviously, mean-square differentiability implies differentiability in probability, but the converse implication is not true. X is convex, then by lemma 1 from [8], there exist nondecreasing stochastic processes $X'_+, X'_- : I \times \Omega \to \mathbb{R}$, called the right and left derivative of X, respectively such that

$$X'_-(u,\cdot) \leqslant X'_+(u,\cdot) \leqslant X'_-(v,\cdot) \leqslant X'_+(v,\cdot) \quad \text{(a.e.).} \tag{3}$$

for all $u, t \in I$, $u < v$.
By the differentiability of X, we have

$$\begin{cases} X'_-(u,\cdot) = X'_+(u,\cdot) = X'(u,\cdot) & \text{(a.e.)} \\ X'_-(v,\cdot) = X'_+(v,\cdot) = X'(v,\cdot) & \text{(a.e.).} \end{cases} \tag{4}$$

By (3) and (4), for all $u, v \in I$, such that $u < v$, we get

$$X'(u,\cdot) \leqslant X'(v,\cdot) \quad \text{(a.e.)} \tag{5}$$

Suppose now, that a mean-square derivative is nondecreasing. It means the inequality (5) holds for all $u, v \in I$, such that $u < v$. Fix $t_0 \in (a,b) \subset I$ and take $t \in (a,b)$, such that $t_0 < t$. By basic properties of mean-square integral (cf. [9]; see also [1]) and the inequality (5) we get

$$X(t,\cdot) - X(t_0,\cdot) = \int_{t_0}^{t} X'(s,\cdot)ds \geqslant \int_{t_0}^{t} X'(t_0,\cdot)ds = X'(t_0,\cdot)(t-t_0) \quad \text{(a.e.).}$$

If $t < t_0$ we receive similarly

$$\begin{aligned} X(t,\cdot) - X(t_0,\cdot) &= \int_{t_0}^{t} X'(s,\cdot)ds = -\int_{t}^{t_0} X'(s,\cdot)ds \geqslant -\int_{t}^{t_0} X'(t_0,\cdot)ds \\ &= X'(t_0,\cdot)(t-t_0) \quad \text{(a.e.).} \end{aligned}$$

It means X has the support of the form

$$X(t,\cdot) \geqslant X(t_0,\cdot) + X'(t_0,\cdot)(t-t_0) \quad \text{(a.e.)}$$

at any point $t_0 \in (a,b)$. Lemma 2.2 completes the proof. □

As before, by Lemma 2.1 and proved above Lemma 2.4, it can be shown the following theorem

Theorem 2.5. Let $X : I \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process. X is strongly convex with modulus $C(\cdot)$ if and only if the first derivative is strongly increasing, that is for all $u, v \in I$, such that $u < v$ the following inequality holds

$$X'(v,\cdot) - X'(u,\cdot) \geqslant 2C(\cdot)(v-u) \quad \text{(a.e.)}. \tag{6}$$

Proof. By the representation of the strongly convex stochastic processes with modulus $C(\cdot)$ (Lemma 2.1), there exists a convex stochastic process $H : I \times \Omega \to \mathbb{R}$, such that

$$X(t,\cdot) = H(t,\cdot) + C(\cdot)t^2 \quad \text{(a.e.)}$$

for all $t \in I$. H is convex and mean-square differentiable, so by Lemma 2.4 the first derivative of H is nondecreasing. For all $u, v \in I$, such that $u < v$ we have

$$H'(u,\cdot) \leqslant H'(v,\cdot) \quad \text{(a.e.)}. \tag{7}$$

It can be easily shown, that the first mean-square derivative of $X(t,\cdot) = H(t,\cdot) + C(\cdot)t^2$ (a.e.) is equal to

$$X'(t,\cdot) = H'(t,\cdot) + 2C(\cdot)t \quad \text{(a.e.)} \tag{8}$$

for all $t \in I$. By (8) for $u, v \in I$ we have

$$\begin{cases} X'(u,\cdot) = H'(u,\cdot) + 2C(\cdot)u & \text{(a.e.)} \\ X'(v,\cdot) = H'(v,\cdot) + 2C(\cdot)v & \text{(a.e.)}. \end{cases} \tag{9}$$

By (7) and (9) we get

$$X'(u,\cdot) - 2C(\cdot)u \leqslant X'(v,\cdot) - 2C(\cdot)v \quad \text{(a.e.)}.$$

It means (6) holds.
Suppose now, that the inequality (6) holds. Let us take

$$H'(t,\cdot) := X'(t,\cdot) - 2C(\cdot)t \quad \text{(a.e.)}$$

for all $t \in I$. By the definition process H is mean-square differentiable. From (6) for fixed $u, v \in I$, such that $u < v$, we get $H'(u,\cdot) \leqslant H'(v,\cdot)$ (a.e.). By Lemma 2.4 H is a convex stochastic process. Now, by Lemma 2.1, we have that

$$X(t,\cdot) = H(t,\cdot) + C(\cdot)t^2 \quad \text{(a.e.)}$$

is a strongly convex with modulus $C(\cdot)$ stochastic process. This completes the proof. □

Finally we present a characterization of strongly convex stochastic processes by use of the second derivative.

Lemma 2.6. Let $X : I\times\Omega \to \mathbb{R}$ be a twice mean-square differentiable stochastic process. X is convex in I if and only if $X''(t,\cdot) \geqslant 0 \quad$ (a.e.) for every $t \in I$.

Proof. Suppose first that a process $X : I \times \Omega \to \mathbb{R}$ is convex. By Lemma 2.4, the first mean-square derivative $X'(t,\cdot)$ is nondecreasing in the interval I. Fix $t, t_0 \in I$ such that $t_0 < t$. By the monotonicity of the first derivative, we have

$$\frac{X'(t,\cdot) - X'(t_0,\cdot)}{t - t_0} \geqslant 0 \quad \text{(a.e.)}.$$

In the case $t < t_0$, we get also

$$\frac{X'(t_0,\cdot) - X'(t,\cdot)}{t_0 - t} \geqslant 0 \quad \text{(a.e.)}.$$

Passing to the mean-square limit, we obtain

$$X''(t_0,\cdot) \geqslant 0 \quad \text{(a.e.)}.$$

Now, let $X''(t,\cdot) \geqslant 0 \quad$ (a.e.) for all $t \in I$. Fix $t_0 \in intI$ and take $t \in I$ such that $t_0 < t$. Calculating the mean-square integral twice, we have

$$0 \leqslant \int_{t_0}^{t} X''(s,\cdot)ds = X'(t,\cdot) - X'(t_0,\cdot) \quad \text{(a.e.)},$$

and

$$\begin{aligned} 0 \leqslant \int_{t_0}^{t} \big[X'(s,\cdot) - X'(t_0,\cdot)\big]ds &= \int_{t_0}^{t} X'(s,\cdot)ds - \int_{t_0}^{t} X'(t_0,\cdot)ds \\ &= X(t,\cdot) - X(t_0,\cdot) - X'(t_0,\cdot)(t - t_0) \quad \text{(a.e.)}. \end{aligned}$$

In the case when $t < t_0$ we have

$$0 \leqslant \int_{t}^{t_0} X''(s,\cdot)ds = X'(t_0,\cdot) - X'(t,\cdot) \quad \text{(a.e.)},$$

and

$$\begin{aligned} 0 \leqslant \int_{t}^{t_0} \big[X'(t_0,\cdot) - X'(s,\cdot)\big]ds &= \int_{t}^{t_0} X'(t_0,\cdot)ds - \int_{t}^{t_0} X'(s,\cdot)ds \\ &= X'(t_0,\cdot)(t_0 - t) - X(t_0,\cdot) + X(t,\cdot) \quad \text{(a.e.)}. \end{aligned}$$

Thus, there exists a support of X the form

$$X(t,\cdot) \geqslant X(t_0,\cdot) + X'(t_0,\cdot)(t - t_0) \quad \text{(a.e.)}$$

for any number $t_0 \in intI$. By Lemma 2.2 X is convex. □

As an immediate consequence of Lemma 2.6 and Lemma 2.1 we get the following theorem.

Theorem 2.7. Let $X : I \times \Omega \to \mathbb{R}$ be a twice mean-square differentiable stochastic process. X is strongly convex with modulus $C(\cdot)$ in I, if and only if $X''(t,\cdot) \geqslant 2C(\cdot) \quad$ (a.e.) for $t \in I$.

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