Mathematica Aeterna, Vol. 6, 2016, no. 4, 561 - 572

Some algebro–geometric aspects of spacetime c–boundary

Vandana
School of Studies in Mathematics, Pt. Ravishankar Shukla
University, Raipur-492010, (C.G.) India
e-mail: vdrai1988@gmail.com

Deepmala
SQC and OR Unit, Indian Statistical Institute,
203 B. T. Road, Kolkata-700 108, India
e-mail: dmrai23@gmail.com

Krzysztof Drachal
Kozminski University, Jagiellońska 57/59,
03–301 Warszawa, Poland
e-mail: kdrachal@kozminski.edu.pl

Vishnu Narayan Mishra[1]
Applied Mathematics and Humanities Department,
Sardar Vallabhbhai National Institute of Technology,
Ichchhanath Mahadev Dumas Road, Surat-395 007, Gujarat, India
e-mail: vishnunarayanmishra@gmail.com

Abstract

We start from the algebraic point of view on a spacetime. In particular, the base structure is not an event represented by a point on a manifold, but the certain algebra of functions. In fact, both approaches are dual to each other – but the second one allows to use algebraic techniques. Next, due to Clarke embedding theorem, we consider the causal boundary of a Lorentzian spacetime (Geroch–Kronheimer–Penrose construction). Then, having algebraic methods already at our disposal, we classify the possible types of singularities.

Mathematics Subject Classification: 58A40, 83C75.

Keywords: causal boundary, generalized manifolds, singularities, spacetime.

[1]Corresponding author

1 Introduction

To use the algebraic approach towards the geometry of spacetime is not only an aesthetic motivation. It comes from physical considerations on the process of observability. For example, in [32] it is widely discussed why algebraic approach can be fruitful. The arguments goes as follows.

First of all, one describes the surrounding reality by certain measurements. Therefore the possessed knowledge about a physical system is a collection of outcomes from some measuring devices. In fact, these measurements are done in certain points x of a manifold M. Therefore, the measuring devices (laboratory) corresponds to some family of real (or complex, or any other) functions. Moreover, this family is an algebra ($\mathbb{R}$–algebra, if real functions are considered). Denote this family by $\mathcal{A}$. In other words, a point is just a state of the physical system and the collection of readings in the given state is just a homomorphism ($\mathbb{R}$–homomorphism) of $\mathcal{A}$ with values in $\mathbb{R}$. (This scheme can be generalized over arbitrary field, but to keep the clarity, we will restrict to real numbers.)

The above scheme can be summarized in the following way: Any manifold M is determined by certain $\mathbb{R}$–algebra $\mathcal{A}$ of functions on this manifold, and every point x on M corresponds to the $\mathbb{R}$–algebra homomorphism $x : \mathcal{A} \to \mathbb{R}$, which assigns to every function $f \in \mathcal{A}$ its value $f(x)$ at the point x.

The above duality does not hold for every space. If it does not hold, it means that there are devices, which introduce no new knowledge about the system; or at least one point is "unobservable". But, actually, this duality works well for manifolds. Of course, this result is well–known and usually connected with Gelfand and Kolmogorov [16]. Sometimes, it is called "Milnor and Stasheff exercise" [30].

It is worth noticing that such research topic dates back to the work of Kähler [25] and it is embedded in the wider context of generalizing differential calculus to commutative algebras (or even more general structures). Such research brought many powerful tools both for pure mathematics and for theoretical physics also [38, 3, 11, 44, 45, 37, 4]. For a deeper and fundamental considerations see, for example, [34]. For a particular example of a cosmological toy model, in which algebraic methods lead to surprising reinterpretations see, for example, [23]. The causality itself on the background of non–commutative structures is discussed, for example, in [13].

Notice also, that there has been various kind of such generalizations. For a review of them see, for example, [1] and references therein. This paper will deal with a particular kind of the Sikorski differential spaces. These spaces seem to be the most straightforward objects, for which the mentioned observability principle can be applied.

Unfortunately, there seems to be little interest in applying such methods to cosmology and general relativity yet. The first, who proposed to reformulate the classical theory in such a way, that the events play no role seems to be Geroch and his approach to quantization of general relativity [14]. Nevertheless, his proposition was later exploited rather by philosophers of science than active physicists or mathematicians [27].

Despite the above facts, the Polish group gathered around Heller and Sasin did much in this direction in 1990s (see [24] and references therein). However, their research was mostly motivated by endeavor to find suitable category, which would allow to describe spacetime with its singularities (in a well–known sense arising from celebrated Penrose–Hawking theorems [22, 35, 21]). Yet, their research direction was initiated directly from Sikorski ideas. Whereas Mallios and his collaborators developed "abstract differential geometry" starting from the theory of Banach spaces and topological algebras. Yet, their differential triads over topological spaces are just another endeavor to build generalized differential calculus [28, 29].

In this paper we will discuss the applicability of methods presented in [8] and [10] to the case of the causal boundary.

2 Differential spaces

Suppose that we are given some set M and a collection of real functions, defined on this set. Denote the family of these functions by $\mathcal{A}_0$ and call them *generators*. Of course, one can demand that these functions are continuous, and as a result some topology, $\tau_{\mathcal{A}_0}$, is obtained on M. Now, a real function f (which domain is M) is called *a local $\mathcal{A}_0$–function*, if for every point $p \in M$, there exists some neighborhood $U \in \tau$ and $g \in \mathcal{A}_0$, such that $f|_U = g|_U$. The set of all possible local $\mathcal{A}_0$–functions on M is denoted by $(\mathcal{A}_0)_M$. Moreover, consider

$$\mathrm{sc}(\mathcal{A}_0) := \{\omega \circ (f_1, \dots, f_n) \mid \omega \in C^\infty(\mathbb{R}^n), f_1, \dots, f_n \in \mathcal{A}_0, n \in \mathbb{N}\} \quad .$$

$\mathrm{sc}(\mathcal{A}_0)$ is called a *superposition closure of* $\mathcal{A}_0$.

Now, suppose that we start from having some set M and a family of real valued functions, defined on this set (denoted by $\mathcal{A}_0$). Suppose, that we first take a closure of this family with respect to the superposition closure, and next we take a closure of a family obtained in such a way, with respect to the localization.

In other words, consider the pair $(M, \mathcal{A})$, where $\mathcal{A} = (\mathrm{sc}\mathcal{A}_0))_M$. $(M, \mathcal{A})$ is called a *differential space* and $\mathcal{A}$ is called a *differential structure on* M, then. If only the superposition closure is demanded (i.e. $\mathcal{A} = \mathrm{sc}\mathcal{A}$), then $(M, \mathcal{A})$ is called a *predifferential space*. Of course, every differential space is predifferential, but not vice verse. It should be noticed that $\tau_{\mathcal{A}_0} = \tau_{\mathrm{sc}\mathcal{A}_0}$ ([7]).

It is known, that if $\mathcal{A} = C^\infty(M)$, then $(M, \mathcal{A})$ is just an infinitely differentiable manifold (in a classical sense). However, the differential calculus and the differential geometry can be constructed over an arbitrary differential space [41, 7, 6]. For example, manifolds with boundaries are a special case of differential spaces.

Generally, one can take an arbitrary "weird" function (for example: non–smooth one) and incorporate it into the initial family of generators $\mathcal{A}_0$. Classically, developing the differential geometry over such a space requires some special techniques at the edges; and in "singular" points. But differential spaces gives unified techniques, and smooth manifolds are just a special subcases of them.

3 Causal boundary

Let M be a spacetime, i.e. infinitely differentiable, 4–dimensional manifold with a Lorentzian metric. For arbitrary points $p, q \in M$ consider the relation $\prec$, defined in the following way: $p \prec q \overset{def.}{\Leftrightarrow}$ there exists a smooth future directed timelike curve from p to q. Then, the *chronological future* of p is defined in the following way: $\mathcal{I}^+(p) := \{q \in M \mid p \prec q\}$. In a similar way, one can define the *chronological past*: $\mathcal{I}^-(p) := \{q \in M \mid p \succ q\}$.

Next, the set $A \subset M$ is called a *future set*, if $\mathcal{I}^+(A) \subseteq A$. Similarly, the set $A \subset M$ is called a *past set*, if $\mathcal{I}^-(A) \subseteq A$. Moreover, if $\mathcal{I}^+(A) = A$, then A is called an *open future set*, and if $\mathcal{I}^-(A) = A$, then A is called an *open past set*.

An open future set, which cannot be expressed as a union of two proper subsets, both of which are open future sets, is called an *indecomposable future set*, or shortly an IF. Similarly, an *indecomposable past set*, or shortly an IP, is defined.

Moreover, if A is an IF and A is not $\mathcal{I}^+(p)$ for any $p \in M$, then such A is called a *terminal indecomposable future set*, or shortly a TIF. Similarly, a *terminal indecomposable past set*, or shortly a TIP, is defined.

Now, consider the collection of all IFs and denote it by $\check{M}$. Similarly, denote all IPs by $\hat{M}$.

Now, in order to keep clarity, let us remind that a spacetime is called *strongly causal*, if it is strongly causal in its every point. *Strong causality in a point* p means that p has an arbitrarily small causally convex neighbourhoods. Whereas an open set U, such that no non–spacelike curve intersects U in a disconnected set, is called a *causally convex*. It is also important to stress, that the set of points of an arbitrary spacetime, at which it is strongly causal is an open subset of this spacetime [2]. (More on causal boundary can be found, for example, in [15, 2, 43].)

It is known, that for an arbitrary strongly causal spacetime, IFs (IPs), which are not TIFs (TIPs), are in one–to–one correspondence with the points of this spacetime [2]. (It can be false, if the spacetime is not strongly causal, as the following example shows: Consider $\mathbb{R} \times S^1$ with the metric $ds^2 = dtd\theta$. Then $W := \{(t, \theta) \mid t < 0\}$ is an IP and suppose that $W = \mathcal{I}^-(\gamma)$ for a future directed and future inextendible timelike curve γ. But W is also a chronological past of an arbitrary point from the circle $\{t = 0\}$.)

In view of the above remarks, let us consider the quotient space $M^\sharp := \hat{M} \cup \check{M}/_\sim$, where the relation $\sim$ identifies $\mathcal{I}^-(p)$ with $\mathcal{I}^+(p)$ for every $p \in M$. Then, consider the map $\mathcal{I}^+ : M \ni p \mapsto \mathcal{I}^+(p) \in M^\sharp$, which identifies M with a subset of $M^\sharp$. As a result, $M^\sharp$ corresponds to M together with all TIPs and TIFs.

The next step is to identify the smallest number of points in $M^\sharp$, as necessary, to obtain the Hausdorff space $M^* = M \cup \partial_c M$. The "additional" points, $\partial_c M$, are called *c–boundary* or *causal boundary*. This topological identification is equivalent to consider M^* as $M^\sharp/_{R_h}$, where R_h is the intersection of all equivalence relations R on $M^\sharp$, such that $M^\sharp/_R$ has the Hausdorff property. In order to ensure that R_h exists, a spacetime has to be strongly causal. Therefore, it is an important problem how to define the topology on $M^\sharp$ and which points should be identified.

Classically, the topology considered on $M^\sharp$ is defined through the following steps. For $A \in \check{M}$, let $A^{int} := \{U \in \hat{M} \mid U \cap A \neq \emptyset\}$ and $A^{ext} := \{U \in \hat{M} \mid U = \mathcal{I}^-(V) \Rightarrow \neg\mathcal{I}^+(V) \subset A\}$. For $B \in \hat{M}$, let $B^{int} := \{U \in \check{M} \mid U \cap B \neq \emptyset\}$ and $B^{ext} := \{U \in \check{M} \mid U = \mathcal{I}^+(V) \Rightarrow \neg\mathcal{I}^-(V) \subset B\}$. Then, $A^{int}, A^{ext}, B^{int}$ and B^{ext} considered for all As and Bs form the subbasis of the topology of $M^\sharp$.

But it is already well–known, that:

Theorem 3.1 ([36]). *The following conditions are equivalent:*

- *M is strongly causal.*
- *The Alexandrov topology induced on M agrees with the one given on the manifold. (The basis of the Alexandrov topology is $\{\mathcal{I}^+(p) \cap \mathcal{I}^-(p) \mid p, q \in M\}$.)*
- *The Alexandrov topology has the Hausdorff property.*

The problem of the topology of causal boundary is not trivial, indeed. Some examples and various discussions can be found, for example, in [12, 18, 26, 19, 20]. Yet, the described construction of the causal boundary has some other drawbacks: there is, in principle, no direct information on singularities, there emerge points at infinity, etc.

Now, let us describe the c–boundary topology construction in the differential spaces formalism. Suppose that we have a differential structure $\mathcal{A}$ on M and a differential structure $\mathcal{A}^\sharp$ in $M^\sharp$. First, we should check, whether the map

$\mathcal{I}^+ : M \ni p \mapsto \mathcal{I}^+(p) \in M^\sharp$ is smooth in the category of differential spaces. In other words ([7]), whether $a^\sharp \circ \mathcal{I}^+ \in \mathcal{A}$ for an arbitrary $a^\sharp \in \mathcal{A}^\sharp$. Therefore, we see that the crucial point is in the relation between differential structures $\mathcal{A}$ and $\mathcal{A}^\sharp$. The smoothness of $\mathcal{I}^+$ can be also seen as a commutativity of the below diagram.

$$\begin{tikzcd} & \mathbb{R} \\ M \arrow[ur, "a^\sharp \circ \mathcal{I}^+"] \arrow[r, "\mathcal{I}^+"'] & M^\sharp \arrow[u, "a^\sharp"'] \end{tikzcd}$$

From the above diagram it is easily seen that if $\mathcal{I}^+$ is smooth in the category of differential spaces, then $\mathcal{A}^\sharp \subset \mathcal{A}$.

For $(M^\sharp, \mathcal{A}^\sharp)$ to posses the Hausdorff property, it is enough that for every two points $m, n \in M^\sharp$, there exists a certain function $a^\sharp \in \mathcal{A}^\sharp$ such that $a^\sharp(m) \neq a^\sharp(n)$. Nevertheless (see, for example, [39] or [7]), even if such a requirement is not fulfilled, then we can consider the following differential space $(M^\sharp/\rho, \mathcal{A}^\sharp/\rho)$, where ρ is an equivalence relation defined in the following way $a^\sharp \, \rho \, b^\sharp \Leftrightarrow \forall_{m \in M^\sharp} \, a^\sharp(m) = b^\sharp(m)$. Therefore, we can assume that $(M^\sharp, \mathcal{A}^\sharp)$ has already the Hausdorff property. Of course, this information is encoded in the differential structure $\mathcal{A}^\sharp$. Another significant role of $\mathcal{A}^\sharp$ is that it contains the information about the topology on $M^\sharp$. In other words, it is the weakest one, for which all functions from $\mathcal{A}^\sharp$ are continuous, i.e., $\tau_{\mathcal{A}^\sharp}$.

Now, the possible problems with the topology on the c–boundary, expressed with a help of $\mathcal{A}^\sharp$, are clearly seen (in a similar way, as in [8]).

Proposition 3.2. *If $\mathcal{I}^+$ is smooth in the category of differential spaces, then exactly one of the below listed cases is possible:*

- *$\mathcal{A}$ contains significantly more functions than constant ones and $\mathcal{A}^\sharp$ consists of only constant functions, i.e, $\mathcal{A}^\sharp \cong \mathbb{R}$,*
- *$\mathcal{A}^\sharp \cong \mathcal{A}$,*
- *$\mathcal{A}^\sharp \ncong \mathcal{A}$, but $(\mathcal{A}^\sharp)_M \cong \mathcal{A}$,*
- *$\mathcal{A}^\sharp \subsetneq \mathcal{A}$.*

Yet, according to [8], the first and the last case correspond to a malicious singularity. The second case corresponds to the c–complete spacetime, i.e., the c–boundary is empty. The third case corresponds to the situation, in which the c–boundary is exactly the topological boundary.

Despite the mentioned problems with c–boundary (both in a classical and presented in this paper approach), its main advantage (which is crucial for the technique described in this paper) is that the causal boundary uses the causal structure of spacetime. As an example, for a flat Minkowski spacetime the future causal boundary is just the future lightlike infinity and timelike infinity. For a Schwarzschild black hole, the causal boundary consists of an additional singularity. Indeed, the causal boundary is invariant under conformal changes of the spacetime. Notice, that, for example, b–boundary (bundle boundary) is not invariant under conformal changes [40, 2].

According to Clarke [5]:

Theorem 3.3. *The spacetime of general relativity can be embedded isometrically in the pseudo–Euclidean space of signature $q-2$, i.e. in $E^{2,q+2}$, where $q=46$, if the spacetime is compact, or $q=87$, if the spacetime is non–compact.*

But, of course, the isometry guarantees that the causal structure is the same. Therefore, instead of constructing the causal boundary of the initial space, we will focus on the causal boundary of the isometrically embedded subspace of the suitable pseudo–Euclidean space. Notice, that this is a subtle argument, because specifying the causal structure and timelike geodesics determines the metric, but the sole causal structure is not enough to determine the metric structure (for a wider context, see, for example, [42]).

4 Main construction

For further considerations, it is important to define diffeomorphisms. This is because we will identify diffeomorphic spaces. In other words we will consider the category, in which differential spaces are the objects and below defined diffeomorphisms are the morphisms in this category.

Definition 4.1. *For two predifferential spaces $(M,\mathcal{A})$ and $(N,\mathcal{B})$, the mapping $F\ : M \to N$ is called* smooth*, if $\forall_{f\in\mathcal{B}}\ f\circ F \in \mathcal{A}$. F is called* diffeomorphism*, if it is bijective and both F and F^{-1} are smooth.*

The generator embedding is a very important example of a diffeomorphism. Let $(M,\mathcal{A})$ be a predifferential space, generated by $f_1,\ldots,f_n$, i.e. $\mathcal{A}=\mathrm{sc}\{f_1,\ldots,f_n\}$, where every $f_i : M \to \mathbb{R}$ and $i=1,\ldots,n$. A *generator embedding* is the mapping defined in the following way:

$$\begin{aligned} F : (M,\mathcal{A}) \to (F(M), C^\infty(\mathbb{R}^n)|_{F(M)}) \quad , \\ M \ni x \mapsto (f_1(x),\ldots,f_n(x)) \in \mathbb{R}^n \quad . \end{aligned}$$

Then $(F(M), C^\infty(\mathbb{R}^n)|_{F(M)})$ is called *the generator image.* Notice that the generators $f_1,\ldots,f_n$ can be understood as coordinates.

Now, let $\mathcal{A}$ be a function $\mathbb{R}$–algebra (an algebra – with respect to pointwise addition and multiplication – of real functions defined on M). Then $\mathrm{Spec}\mathcal{A} := \{\chi : \mathcal{A} \to \mathbb{R} \mid \chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{R}), \chi(\mathbb{1}) = 1\}$. $\mathrm{Spec}\mathcal{A}$ is called the *spectrum of an algebra* $\mathcal{A}$. As it has been discussed previously, "good" spaces are characterized by a suitable correspondence between M and $\mathrm{Spec}\mathcal{A}$.

Definition 4.2. *Consider mappings* $\widehat{f} : \mathrm{Spec}\mathcal{A} \to \mathbb{R}$*, such that* $\widehat{f}(\chi) := \chi(f)$*. The weakest topology on* $\mathrm{Spec}\mathcal{A}$*, for which* $\widehat{f}$ *are continuous for every* $f \in \mathcal{A}$*, is called the* Gelfand topology*. The collection of all such mappings is denoted by* $\widehat{\mathcal{A}}$*, i.e.* $\widehat{\mathcal{A}} := \{\widehat{f} : \mathrm{Spec}\mathcal{A} \to \mathbb{R} \mid f \in \mathcal{A}\}$*.*

Actually, it is not hard to prove that if $(M, \mathcal{A})$ is a predifferential space, then $(\mathrm{Spec}\mathcal{A}, \widehat{\mathcal{A}})$ is also a predifferential space (sometimes called a *spectral space*). Indeed, the topology on a spectral space is introduced in the same manner as on a predifferential space. The superposition closure is an easy calculation, using the condition $\widehat{f}(\chi) := \chi(f)$. It is also not hard to check localization closure.

Now, we can ask, whether $(M, \mathcal{A})$ and $(\mathrm{Spec}\mathcal{A}, \widehat{\mathcal{A}})$ are diffeomorphic. The answer is positive for differential spaces, but for predifferential spaces the answer is more subtle. In particular:

Theorem 4.3. $(M, \mathcal{A})$ *and* $(\mathrm{Spec}\mathcal{A}, \widehat{\mathcal{A}})$ *are diffeomorphic, if and only if the generator image is closed with respect to the (pseudo–)Euclidean metric.*

Proof. The detailed proof is rather technical and quite long, therefore it is presented in [9].

The idea is the following:

If M is closed in $\mathbb{R}^n$, then it can be shown that the smooth structure is generated just by superposition closure of projections on all n axises. In other words, that the localizations closure can be omitted; or that a predifferential space is a differential space in such a case.

If M is not closed in $\mathbb{R}^n$, then certain function is constructed and its Gelfand representation (in a sense of Def. 4.2) is computed. Also, certain sequence of diffeomorphic spaces is constructed. □

Therefore the topological boundary can be expressed in terms of spectra as $\mathrm{Spec}\mathcal{A} \backslash \mathrm{Spec}\mathcal{A}_M$. Notice, that this representation is unique only up to a diffeomorphism, because no metric structure is yet defined on spectral space. But it is possible to lead these considerations further and obtain the isometry. The technical details will be presented elsewhere (see also [8]).

Now, we will present application of the just presented theory to the problem of a causal boundary of a spacetime. Let M be a spacetime. In other words, it is the differential space $(M, C^\infty(M))$ with an additional structure: a Lorentzian metric g. For sufficiently large $n \in \mathbb{N}$, according to Th. 3.3, M

can be isometrically embedded in $\mathbb{R}^n$ with pseudo–Euclidean metric. In other words, it is embedded in the differential space $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ with the standard pseudo–Euclidean metric. Because this embedding is an isometry, then both differential spaces have the same causal structure, and as mentioned before, they have "equivalent" causal boundaries.

The details can be express in the following way: Th. 3.3 guarantees the existence of the generator embedding

$$F : (M, C^\infty(M)) \to (F(M), C^\infty(\mathrm{sc}\{\pi_1, \ldots, \pi_n\})_{F(M)}) \quad ,$$

where π_i, $i = 1, \ldots, n$, are the projections on i-th axis. In particular, the finite collection of functions $f_1, \ldots, f_n$, where $F = (f_1, \ldots, f_n)$ is obtained. Clearly, $\{\pi_i^*(f_i) := f_i \circ \pi_i \mid i = 1, \ldots, n\}$ are the generators of $C^\infty(M)$.

Now, it is clear that $\partial_c M$ is the causal boundary of the Lorentzian manifold. Unfortunately, we cannot assume that

$$\partial_c M \cong \partial_{gen} M \overset{def.}{=} \mathrm{Spec}\mathcal{A} \backslash \mathrm{Spec}\mathcal{A}_M \quad ,$$

where $\mathcal{A} = \mathrm{sc}\{f_1, \ldots, f_n\}$ and $\mathcal{A}_M = C^\infty(M)$, as it was done for the case of b–boundary in [10, 8]. However, with a help of Prop. 3.2, we can switch our consideration to the relationship between $\mathcal{A}$ and $\mathcal{A}^\sharp$, which induces four types of c–boundary.

ACKNOWLEDGEMENTS. K.D. would like to thank Erwin Schrödinger International Institute for Mathematical Physics of the University of Vienna (ESI) for the support during "Summer School on Mathematical Relativity" (2014) and acknowledge that partially his research was funded by the Polish National Science Centre grant under the contract number DEC–2012/06/A/ST1/00256.

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Received: July, 2016