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Singularity formation of compressible Euler equations with source term

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Abstract

In this paper, we mainly study the blow-up of classical solutions to one-dimensional compressible Euler equations with source term. We discussed separately the cases of source term with Riemann invariants and Riemann invariants function. The method we adopted here is to reduce the system to a diagonal form system by introducing the Riemann invariant. Then, we proved the solution blow-up in a finite time when the initial data satisfy certain conditions.

Mathematics Subject Classification: xxxxx

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1 Introduction

We are interested in the blow-up of classical solutions to the Cauchy problem of one-dimensional compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x + \dfrac{1}{\rho} p_x = 0, \\ \rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \end{cases} \tag{1.1}$$

where x is the spatial variable, $t \in \mathbb{R}^+ = [0, \infty)$ is the time. u stands for the velocity of the gas in the x direction. ρ, p denote the density and pressure, respectively. Besides, $p = K\rho^\gamma$ (constants $K > 0, \gamma > 1$), $\rho_0(x) > \overline{\rho} > 0$ ($\overline{\rho}$ is a positive constant), $\rho_0(x) \in C^1(\mathbb{R})$, $u_0(x) \in C^1(\mathbb{R})$. We study the singularity formation of compressible Euler equations (1.1) with the source term of both Riemann invariants and Riemann invariants function.

Before proceeding, we briefly review some previous results of singularity formation for compressible Euler equations (1.1). In the case of one dimension, the relativistic methods have been treated extensively. [5, 6, 13] obtained that no matter how small and smooth the initial value is, the solution of (1.1) will blow-up in finite time. The blow-up for compressible Euler equations under Lagrangian coordinates has studied by [1]. For more information about this model, we can refer to the review [3, 4]. In the case of more than one space dimension, the earlier study was [12], in this paper, author used the method via certain averaged quantities to prove the formation of singularities in three-dimensional compressible Euler equations. Besides, [8,10,14] and [2,11] used a similar approach to obtain other formation of singularity theorem for classical fluids and relativistic fluids, respectively. [9] studied the blow-up of smooth solutions for the relativistic Euler equations. But none of above studied the singularity formation of compressible Euler equations with the source term like our paper does.

2 Main Results

To our end, we first introduce the system of compressible Euler equations with the source term of Riemann invariants

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x + \dfrac{1}{\rho} p_x = u - \dfrac{2c}{\gamma - 1}, \\ \rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \end{cases} \tag{2.1}$$

where ρ, u, p denote the same thing as above, define $c^2 = \dfrac{dp}{d\rho}$. We introduce the Riemann invariant $w = u - \dfrac{2c}{\gamma - 1}$, $z = u + \dfrac{2c}{\gamma - 1}$, $w|_{t=0} = w_0$, $z|_{t=0} = z_0$.

Remark 2.1 *When the system* (1.1) *has the source term of Riemann invariants, we only study the case where the source term is* $w = u - \dfrac{2c}{\gamma - 1}$. *In the case of* $z = u + \dfrac{2c}{\gamma - 1}$, *the research method is the same, we won't go into details here.*

The first main result of this paper is given below.

Theorem 2.2 *Suppose that at least one of the equations w_0 and z_0 is not a monotonic increasing function, then the solution of system* (2.1) *will form singularity in finite time.*

Then we introduce a more general system of compressible Euler equations with the source term of Riemann invariants function

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x + \dfrac{1}{\rho} p_x = f(u - \dfrac{2c}{\gamma - 1}) + M, \\ \rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \end{cases} \tag{2.2}$$

where $f(u - \dfrac{2c}{\gamma - 1}) \in C^1(\mathbb{R})$ and bounded, M is a constant.

Remark 2.3 *Similarly, we only study the case where the source term is $f(u - \dfrac{2c}{\gamma - 1}) + M$, for the source term $f(u + \dfrac{2c}{\gamma - 1}) + N$ (N is a constant), the research method is the same, we won't repeat it here.*

The second main theorem of this paper is the following.

Theorem 2.4 *Suppose that at least one of the equations w_0 and z_0 is not a monotonic increasing function, $f(w)\big(f(z)\big) \in C^1(\mathbb{R})$ and bounded, then the solution of system* (2.2) *will form singularity in finite time.*

3 Proof of Theorem 2.2

Define

$$U = \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad A = \begin{bmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{bmatrix},$$

then the first two equation of (2.1) can be rewrite as

$$\partial_t U + A \partial_x (U) = \begin{bmatrix} 0 \\ u - \frac{2c}{\gamma - 1} \end{bmatrix}.$$

The eigenvalue of A are

$$\lambda = u - c, \quad \mu = u + c.$$

The corresponding left eigenvectors are

$$\xi_1 = (c, -\rho), \quad \xi_2 = (c, \rho), \tag{3.1}$$

right eigenvectors are

$$\zeta_1 = (\rho, -c)^T, \quad \zeta_2 = (\rho, c)^T. \tag{3.2}$$

Recall the Riemann invariants

$$\begin{aligned} w &= u - \int \frac{c}{\rho} d\rho = u - \frac{2c}{\gamma - 1}, \\ z &= u + \int \frac{c}{\rho} d\rho = u + \frac{2c}{\gamma - 1}, \end{aligned} \tag{3.3}$$

thus the first two equations of (2.1) can be rewrite as

$$\begin{cases} \partial_t w + \lambda(w,z)\partial_x w = w, \\ \partial_t z + \mu(w,z)\partial_x z = w, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \lambda(w,z) = u - c = \dfrac{\gamma+1}{4} w + \dfrac{3-\gamma}{4} z, \\ \mu(w,z) = u + c = \dfrac{3-\gamma}{4} w + \dfrac{\gamma+1}{4} z. \end{cases} \tag{3.5}$$

Define

$$\acute{} = \partial_t + \lambda \partial_x \qquad \grave{} = \partial_t + \mu \partial_x.$$

Firstly, we consider w, take the derivative of both sides of the equation $(3.4)_1$ with respect to x, we have

$$w_{tx} + \lambda w_{xx} + \lambda_w w_x^2 + \lambda_z w_x z_x = w_x. \tag{3.6}$$

Besides, from

$$\begin{aligned} w = z_t + \mu z_x &= z\grave{}, \\ z_t + \lambda z_x &= z\acute{}, \end{aligned} \tag{3.7}$$

we have

$$z_x = \frac{z\acute{} - w}{\lambda - \mu}. \tag{3.8}$$

Then, combining (3.6) and (3.8), we have

$$w_{tx} + \lambda w_{xx} + \lambda_w w_x^2 + \lambda_z w_x \frac{z\acute{} - w}{\lambda - \mu} = w_x. \tag{3.9}$$

Define $\alpha = w_x$, we obtain

$$\alpha\acute{} + \lambda_w \alpha^2 + \alpha(\tfrac{z\acute{} - w}{\lambda - \mu}\lambda_z - 1) = 0. \tag{3.10}$$

Define h is a function of w, z and satisfy

$$-h_w = h_z = \frac{\lambda_z}{\lambda - \mu}, \tag{3.11}$$

that is $h(w,z) = \dfrac{\gamma-3}{2(\gamma-1)}\ln(z-w)$.

From $w\prime = w$, we have

$$h\prime = h_w w\prime + h_z z\prime = -\frac{\lambda_z}{\lambda-\mu} w + \frac{\lambda_z}{\lambda-\mu} z\prime = \frac{z\prime - w}{\lambda-\mu}\lambda_z. \tag{3.12}$$

Thus, combining (3.10) and (3.12), we obtain

$$\alpha\prime + \lambda_w \alpha^2 + \alpha(h\prime - 1) = 0. \tag{3.13}$$

Multiply both sides of equation (3.13) by e^{h-t}, we have

$$\alpha\prime e^{h-t} + \alpha(h\prime - 1)e^{h-t} + \lambda_w \alpha^2 e^{h-t} = 0, \tag{3.14}$$

that is

$$(\alpha e^{h-t})\prime + \lambda_w e^{-h+t}(\alpha e^{h-t})^2 = 0, \tag{3.15}$$

where we have used

$$(h-t)\prime = h\prime - 1. \tag{3.16}$$

Define

$$\widetilde{\alpha} = \alpha e^{h-t},$$

thus

$$\widetilde{\alpha}\prime = -a\widetilde{\alpha}^2, \tag{3.17}$$

where

$$a = \lambda_w e^{-h+t} = \frac{\gamma+1}{4}(z-w)^{\frac{3-\gamma}{2(\gamma-1)}} e^t > 0. \tag{3.18}$$

From the definition of α, we can get it's initial value α_0 satisfy

$$\widetilde{\alpha}(x_0, 0) = e^{h(w_0(x), z_0(x))}\alpha_0(x). \tag{3.19}$$

Suppose $x = x(x_0, t)$ is the first characteristic curve that starts at any point $(x_0, 0)$, then integrate the equation (3.17) along it to obtain

$$\frac{1}{\widetilde{\alpha}(x(t),t)} = \frac{1}{\widetilde{\alpha}(x_0,0)} + \int_0^t a(x(v),v)dv. \tag{3.20}$$

Suppose that at least one of the equations w_0 and z_0 is not a monotonic increasing function, from (3.20), we can obtain the Cauchy problem (2.1) will blow-up in a finite time. Thus we have deduce Theorem 2.2.

4 Proof of Theorem 2.4

Similar with the prove of Theorem 2.2, firstly, we define

$$V=\begin{bmatrix}\rho\\ u\end{bmatrix},\quad \Lambda=\begin{bmatrix}u & \rho\\ \frac{c^2}{\rho} & u\end{bmatrix},$$

then the first two equations of (2.2) can be rewrite as

$$\partial_t V+\Lambda\partial_x(V)=\begin{bmatrix}0\\ f(u-\frac{2c}{\gamma-1})+M\end{bmatrix}. \tag{4.1}$$

Similarly, the eigenvalue of Λ are

$$\lambda=u-c,\quad \mu=u+c, \tag{4.2}$$

left eigenvectors are

$$\xi_1=(c,-\rho),\quad \xi_2=(c,\rho), \tag{4.3}$$

right eigenvectors are

$$\zeta_1=(\rho,-c)^T,\quad \zeta_2=(\rho,c)^T. \tag{4.4}$$

Using the Riemann invariants, the first two equations of (2.2) also can be rewrite as

$$\begin{cases}\partial_t w+\lambda(w,z)\partial_x w=f(w)+M,\\ \partial_t z+\mu(w,z)\partial_x z=f(w)+M,\end{cases} \tag{4.5}$$

where

$$\begin{cases}\lambda(w,z)=u-c=\dfrac{\gamma+1}{4}w+\dfrac{3-\gamma}{4}z,\\ \mu(w,z)=u+c=\dfrac{3-\gamma}{4}w+\dfrac{\gamma+1}{4}z.\end{cases} \tag{4.6}$$

Firstly, we consider w, take the derivative of both sides of the equation $(4.5)_1$ with respect to x, we have

$$w_{tx}+\lambda w_{xx}+\lambda_w w_x^2+\lambda_z w_x z_x=f_w(w)w_x. \tag{4.7}$$

From (4.5), we have

$$\begin{aligned}f(w)+M=z_t+\mu z_x&=z\backslash,\\ z_t+\lambda z_x&=z/,\end{aligned} \tag{4.8}$$

thus

$$z_x=\frac{z/-(f(w)+M)}{\lambda-\mu}. \tag{4.9}$$

Combing (4.7) and (4.9), we have

$$w_{tx} + \lambda w_{xx} + \lambda_w w_x^2 + \lambda_z w_x \frac{z\prime - (f(w) + M)}{\lambda - \mu} = f_w(w) w_x, \tag{4.10}$$

from $\alpha = w_x$, we have

$$\alpha\prime + \lambda_w \alpha^2 + \alpha(\frac{z\prime - (f(w) + M)}{\lambda - \mu}\lambda_z - f_w(w)) = 0. \tag{4.11}$$

Define h is a function of w, z and satisfy

$$-h_w = h_z = \frac{\lambda_z}{\lambda - \mu}, \tag{4.12}$$

thus we have $h(w, z) = \dfrac{\gamma - 3}{2(\gamma - 1)} \ln(z - w)$.

Besides, from $w\prime = f(w) + M$, we have

$$h\prime = h_w w\prime + h_z z\prime = -\frac{\lambda_z}{\lambda - \mu}(f(w) + M) + \frac{\lambda_z}{\lambda - \mu} z\prime = \frac{z\prime - (f(w) + M)}{\lambda - \mu}\lambda_z. \tag{4.13}$$

Combing (4.11) and (4.13), we obtain

$$\alpha\prime + \lambda_w \alpha^2 + \alpha(h\prime - f_w(w)) = 0. \tag{4.14}$$

Multiply both sides of equation (4.14) by $e^{h-\ln(f(w)+M)}$, we have

$$\alpha\prime e^{h-\ln(f(w)+M)} + \alpha(h\prime - f_w(w))e^{h-\ln(f(w)+M)} + \lambda_w \alpha^2 e^{h-\ln(f(w)+M)} = 0, \tag{4.15}$$

then we have

$$(\alpha e^{h-\ln(f(w)+M)})\prime + \lambda_w e^{-h+\ln(f(w)+M)}(\alpha e^{h-\ln(f(w)+M)})^2 = 0, \tag{4.16}$$

where we have used

$$(h - \ln(f(w) + M))\prime = h\prime - \frac{(f(w))\prime}{f(w) + M} = h\prime - \frac{f_w(w)w\prime}{f(w) + M} = h\prime - f_w(w). \tag{4.17}$$

Define

$$\widetilde{\alpha} = \alpha e^{h-\ln(f(w)+M)}, \tag{4.18}$$

thus

$$\widetilde{\alpha}\prime = -a_1 \widetilde{\alpha}^2, \tag{4.19}$$

where

$$a_1 = \lambda_w e^{-h+\ln(f(w)+M)} = \frac{\gamma + 1}{4}(z - w)^{\frac{3-\gamma}{2(\gamma-1)}}(f(w) + M), \tag{4.20}$$

because there exist a constant M satisfy $f(w)+M>0$, thus we have $a_1>0$.

From the definition of α, we can get it's initial value α_0 satisfy

$$\widetilde{\alpha}(x_0,0)=e^{h(w_0(x),z_0(x))+\ln(f(w_0(x))+M)}\alpha_0(x). \tag{4.21}$$

Suppose $x=x(x_0,t)$ is the first characteristic curve that starts at any point $(x_0,0)$, then integrate the equation (3.17) along it to obtain

$$\frac{1}{\widetilde{\alpha}(x(t),t)}=\frac{1}{\widetilde{\alpha}(x_0,0)}+\int_0^t a_1(x(v),v)dv. \tag{4.22}$$

Suppose that at least one of the equations w_0 and z_0 is not a monotonic increasing function, from (4.22), we can obtain the Cauchy problem (2.2) will blow-up in a finite time. Theorem 2.4 is proved.

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