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Sequentially Contractions in Cone Metric Spaces

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Abstract

In this paper we study sequentially contractions and prove some theorems on the existence of fixed points for sequentially contractions in cone metric spaces .Also, we obtain an extension of the theorem2.6 of [1].

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1.Introduction

The concept of cone metric space was introduced by Huan Long-Guang and Zhang Xian[2], where the set of real numbers is replaced by an ordered Banach space. They introduced the basic definitions and some properties of convergence of sequences in cone metric spaces. some other mathematicians(for instance [3,4,5]) have generalized the results of Guang and Zhang.

Recently, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh introduced a new class of contractive mappings: T-contraction and T-contractive extending the Banach's contraction principle and the Edelstein's fixed point theorem,(see[6]) respectively. Authors in[7] completely discussed on the fixed points of a TR-contraction.

In this work, we analyze the existence of fixed points for a new class of contractive mappings, so-called sequentially contraction.

We recall the definitions of cone metric space and other results that will be needed in the sequel.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if :

(P1)P is nonempty, closed and $P \neq 0$;

(P2)a, $b \in \mathbb{R}$, a,$b \geq o$ and x,$y \in$ P$\Longrightarrow$ ax+by$\in P$;

(P3)$x \in P \; and \; -x \in P \Rightarrow x = 0.$

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by

$x \leq y \iff y - x \in P$

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stands for $y - x \in IntP$, where $IntP$ denotes the interior of P .the cone $P \subset E$ is called normal if there is a number K >0 such that for all $x, y \in E,$

$0 \leq x \leq y$ *implies* $\|x\| \leq K\|y\|$

The least positive number satisfying inequality above is called the normal constant of P .

Throughout this paper, we always suppose E is a Banach space, is a cone with $IntP \neq \phi$ and $\leq$ is a partial ordering with respect to P .

Definition1.1([2]). Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies:

$0 <$ $d x, y$

(i) for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y;$

$d x$ y $d y$ x

(ii) for all $x, y \in X$;

$d(x, y) \leq d(x, z) + d(z, y)$

(iii) for all $x, y, z \in X$.

Then is called a cone metric on X and (X, d) is called a cone metric space.

Example1.2([2],example1). Let $E = \mathbf{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2, X = \mathbf{R}$ and $d: X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

Where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

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Definition 1.3 ([2]). Let (X,d) be a cone metric space. Let (x_n) be a sequence in X. Then:

(i) (x_n) converges to $x \in X$ if, for every $c \in E$, with $c \gg 0$ there is $n_0 \in N$ such that for all $n \geq n_0$,

$d(x_n, x) \ll c.$

We denote this by $\lim_{n\to\infty} x_n = x$ or or $x_n \to x$, $(n \to \infty).$

(ii) If for any $c \in E$, there is a number $n_0 \in N$ such that for all $m, n \geq n_0$

$d(x_n, x_m) \ll c.$

Then (x_n) is called a Cauchy sequence in X;

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent in X.

Definition 1.4 ([1]). Let (X,d) be a cone metric space, a normal cone with normal constant K and $T : X \to X$. Then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} Tx_n = Tx$ for every (x_n) in X.

(ii) T is said to be sequentially convergent if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) also is convergent.

(iii) T is said to be subsequentially convergent if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) has a convergent subsequence.

For more details see [2,7].

2. Main Results

In this section , first we introduce the notation of sequentially contraction , then we prove some theorem and results .

Definition 2 . 1 . Let (X,d) be a cone metric space and $S:X\to X$ a mapping .

A mapping S is said to be sequentially contraction if there exist $\alpha\in[0,1)$ constant and a sequence (T_n) from X into X , such that S is a T_n – contraction for each $n\in N$, i.e , for each $n\in N$,

$d(T_nSx\,,T_nSy)\le\alpha d(T_nx\,,T_ny)$

For all $x,y\in X$.

Example 2 . 2 . Let $E=(C[0,1],R)$, , $X=R$ and $d(x,y)=|x-y|e^t$,

Where $e^t\in E$. Then (X,d) is a cone metric space. Consider the function $S:X\to X$ by $Sx=x+1$. Also, we consider $T_n:X\to X$ defined by $T_nx=e^{-nx}$, for each $n\in N$. Then:

$d(T_nSx\,,T_nSy)=|e^{-n(x+1)}-e^{-n(y+1)}|e^t=\frac{1}{e}|e^{-nx}-e^{-ny}|e^t=\frac{1}{e}d(T_nx\,,T_ny).$

Where So is sequentially contraction.

Note 2. 3. If $S:X\to X$ is sequentially contraction , then need not not be a contraction ; for instance , in example 2 . 2 , S isn't a contraction ; since

$d(Sx\,,Sy)=|x-y|e^t>\alpha|x-y|e^t=\alpha d(x,y)$

For all $0\le\alpha<1$.

The next theorem extends the theorem 2. 6 of [1].

Theorem 2 . 4 . Let (X,d) be a complete cone metric space , P be normal cone with normal constant K , in addition let $S:X\to X$ be sequentiall contraction with a sequence (T_n) . Then for $x_0\in X$,

$\lim_{n,m\to\infty} d(T_k S^n x, T_k S^m x_0) = 0$

For all $k \in N$.

Proof . Let $x_0 \in X$ and (x_n) given by $x_{n+1} = Sx_n = S^n x_0$, $n = 0,1,\dots$. Notice that

$d(T_k x_n, T_k x_{n+1}) = d(T_k S^{n-1} x_0, T_k S^n x_0) \leq \alpha d(T_k S^{n-2} x_0, T_k S^{n-1} x_0)$

For all $k \in N$. Hence, recursively we obtain

$d(T_k S^{n-1} x_0, T_k S^n x_0) \leq \alpha^{n-1} d(T_k x_0, T_k S x_0)$

For all $k \in N$.

Since P is a normal cone with normal constant k , we get

$$\|d(T_k S^{n-1} x_0, T_k S^n x_0)\| \leq \alpha^{n-1} k \|d(T_k x_0, T_k S x_0)\|$$

Which , taking limits , implies that

$\lim_{n\to\infty} d(T_k S^{n-1} x_0, T_k S^n x_0) = 0$ (2 . 4 . 1)

For all $k \in N$.

Now , Let $m, n \in N$ with $m > n$. Then

$d(T_k S^{n-1} x_0, T_k S^{m-1} x_0) \leq d(T_k S^{n-1} x_0, T_k S^n x_0) + \cdots + d(T_k S^{m-2} x_0, T_k S^{m-1} x_0)$

If $m - n$ is odd , and

$d(T_k S^{n-1} x_0, T_k S^{m-1} x_0) \leq d(T_k S^{n-1} x_0, T_k S^n x_0) + \cdots + d(T_k S^{m-2} x_0, T_k S^{m-1} x_0)$

For $m - n$ even . Since for any $n \in N$ can be oroved analogously to (2 . 4 . 1) that

$d(T_k S^n x_0, T_k S^n x_0) \to 0, as \quad n \to \infty$ (2 . 4 . 2)

For all $k \in N$. Then by (2 .4 . 1) and (2 . 4 . 2) we have

$\lim_{n\to\infty} d(T_k S^{n-1} x_0, T_k S^{m-1} x_0) = 0$

For all $k \in N$, which proves the assertion . □

The following result is the localization of theorem 2 .4.

Theorem 2 . 5 . Let (X,d) be a complete cone metric space , $P \subset E$ be a normal cone with normal constant K and $S : X \to X$ be sequentially contraction with a sequence (T_n). Also , Let S be continuous mapping for all $x, y \in B(T_k x_0 , c)$ and

$$d(T_k S x_0 , T_k x_0) \leq (1-\alpha)c \qquad (k \in N)$$

Where $c \in E$ with $c \gg 0$, $x_0 \in X$ and

$$B(T_k x_0 , c) = \{y \in X : d(T_k x_0 , y) \leq c \quad , k \in N\}$$

Then for each $k \in N$ S has a unique fixed point in $B(T_k x_0 , c)$.

Proof . We only need to prove that $B(T_k x_0 , c)$ is complete for any $k \in N$. Suppose $k_0 \in N$. Let $(y_n) \subset B\left(T_{k_0} x_0 , c\right)$ be a Cauchy sequence. By the completeness of X , there exist $y \in X$ such that $y_n \to y$, $(n \to \infty)$. Thus we have

$$d\left(T_{k_0} x_0 , y\right) \leq d\left(y_n , T_{k_0} x_0\right) + d(y_0 , y) \leq c + d(y_n , y)$$

Since $y_n \to y$, $(n \to \infty)$, So $d\left(T_{k0} x_0 , y\right) \leq c$. Hence $\in B\left(T_{k_0} x_0 , c\right)$.

Therefore $B\left(T_{k_0} x_0 , c\right)$ is complete. Since $k_0 \in N$ was arbitrary, so $B(T_k x_0 , c)$ is complete, for any $k \in N$. □

In the following by $\mathcal{F}$ we mean the family of mappings whose members are either contractive , nonexpansive or α - contraction $(0 < \alpha < 1)$ mappings .

Theorem 2 . 6 . Let (X,d) be a complete cone metric space , P be a normal cone with normal constant K , in addition let $S : X \to X$ be sequentially contraction with a sequence (T_n) . Also , Let S be continuous and (T_n) be one to one and continuous mapping in $\mathcal{F}$. Then for every $x_0 \in X$, the iterate sequence $(S^n x_0)$ converges .

Proof . Let $x_0 \in X$ and $(S^n x_0)$ be as follow

$x_{n+1} = Sx_n = S^n x_0 \qquad n = 0,1, \dots \dots$

If $(S^n x_0)$ does not converges , then for each $n \in N$

$\lim_{n\to\infty} d(S^n x_0 , S^n x_0) \nrightarrow 0 \quad (n \to \infty) \qquad (2.6.1)$

On the other hand , notice that

$d(T_1k\ S^\uparrow n\ x_1 0\ , T_1 k\ S^\uparrow m\ x_\downarrow \mathbf{0}\ ) \leq d(T_1 k\ S^\uparrow n\ x_\downarrow \mathbf{0}\ , T_1 k\ S^\uparrow (n+1)\ x_1 \mathbf{0}\ ) + d(T_1 k\ S^\uparrow (n+1)\ x_\downarrow \mathbf{0}\ , T_1 k\ S^\uparrow (m+1)\ x$

For all $k \in N$.

Then

$$d(T_k S^n x_0 , T_k S^m x_0) \leq \frac{1}{1 - \alpha[d(T_k S^n x_0 , T_k S^{n+1} x_0) + d(T_k S^{m+1} x_0 , T_k S^m x_0)]}$$

Since for any $k \in N$, T_k is in the family $\mathcal{F}$, then inequality above can be rewrite as

$$d(T_k S^n x_0 , T_k S^m x_0) \leq \frac{1}{1 - \alpha[\beta d(S^n x_0 , S^{n+1} x_0) + \gamma d(S^{m+1} x_0 , S^m x_0)]}$$

Where . By (2 . 6 . 1) we can conclude that

$\lim_{n\to\infty} d(T_k S^n x_0 , T_k S^m x_0) \nrightarrow 0 \quad (m \to \infty)$

Which is a contradiction with Theorem 2.4. Therefore, we have that there is $v_0 \in X$ such that $\lim_{n\to\infty} S^n x_0 = v_0$. □

Note 2. 7. One can prove that the sequence $(S^n x_0)$ above converges to the fixed point of S .

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