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Scalar conservation law with discontinuous flux – thickened entropy conditions and doubling of variables

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Abstract

We propose new entropy type conditions for the scalar conservation law with discontinuous flux and prove existence and uniqueness of appropriate entropy solutions.

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1 Introduction

We consider the following problem:

$$\begin{cases} \partial_t u + \partial_x \left(H(x)f(u) + H(-x)g(u) \right) = 0, & (x,t) \in I\!R \times I\!R^+, \\ u|_{t=0} = u_0(x) \in L^\infty(I\!R), & x \in I\!R, \end{cases} \tag{1}$$

where H is the Heaviside function.

We assume that the functions f and g are non-negative, continuously differentiable in the interval $[0,1]$, and such that $f(0)=f(1)=g(0)=g(1)=0$. For the initial condition u_0, we assume that $0 < u_0(x) < 1$, $x \in \mathbf{R}$.

Equations like (1) have received a considerable amount of attention in recent past since they occur in a variety of applications, including flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, and gas flow in a variable duct.

Assume, for instance, that we want to describe oil flow in a heterogeneous media (so called two phase flow) or traffic flow on a road on which conditions are not homogeneous. Then, the unknown function $u(x,t)$ represents oil saturation (or car concentration) at the moment t and the point x. Roughly speaking, if $u(x,t) = 0$ then there is no oil at the point x (or no cars at the point of the road) while if $u(x,t) = 1$ the saturation is maximal (or the road is filled with cars). Therefore, it is standard to assume that the functions $g(u)$ and $f(u)$ are zero if $u \notin (0,1)$ (or some other finite interval since there is no sense to consider equation for values that u can not possibly reach).

Problem (1) is not interesting only from the viewpoint of applications but also from purely mathematical reasons. It appeared that the it is rather complex and that it is not possible to directly generalize methods from the theory of scalar conservation law with smooth flux [10].

In the case of a scalar conservation law with smooth flux, the existence was proved by the shifting of variables method applied on a sequence of solutions to the corresponding Cauchy problem regularized with the vanishing viscosity. The method does not give results if the flux is discontinuous. Therefore, we need to apply more subtle arguments involving singular mapping [15], compensated compactness [9] or H-measures [2, 14].

Even more difficult is the question of uniqueness of a weak solution to (1). As well known, considered problem has several weak solutions satisfying the same initial condition. Therefore, we have to find additional conditions which should be satisfied by a weak solution in order to single out the proper one (admissible one).

Many different criterions were proposed in the past. We mention minimal variation criterion [6], Γ condition [5], entropy conditions [8], vanishing capillary pressure limit [7], admissibility conditions via adapted entropies [3], geometric type conditions [1]. In every of the mentioned approach, the assumption on existence of traces for an admissible solution plays a crucial role. Still, the question of existence of traces is very complicated in itself (see e.g. [11, 13]).

Successful attempt on settling the uniqueness and existence issue for a special case of (1) without using traces or a non-degeneracy condition was made in [4]. Their proof is based on the kinetic formulation of conservation laws [12].

A question posed there is whether it is possible to prove existence and uniqueness for (1) by using only Kruzhkov's technique of doubling of variables. Here, also with no hypothesis of convexity or genuine nonlinearity on g and f, we give a positive answer on that question.

2 Uniqueness and Existence of the Admissible Weak Solution

Before we introduce a definition of admissibility, we will try to motivate it.

Take the following step function $k(x) = \begin{cases} k_L, & x > 0 \\ k_R, & x \leq 0 \end{cases}$ for positive constants k_L and k_R, and consider the following special situation of (1) (this is the equation considered in [4]):

$$\partial_t u + \partial_x k(x) g(u) = 0, \tag{2}$$

where g is taken from (1), and we additionally assume that g is convex on $(0,1)$.

Then, assume that $k_L > k_R > 0$ and consider (1) as the following Riemann problem:

$$\begin{aligned} &k_t = 0, && \partial_t u + \partial_x (k g(u)) = 0, \\ &k|_{t=0} = \begin{cases} k_L, & x < 0 \\ k_R, & x > 0, \end{cases} && u|_{t=0} = u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \end{aligned} \tag{3}$$

Also, assume that $0 < u_L, u_R < 1$. We look for the weak solution of (3) which is admissible in the Lax sense. Note that there could be several admissible weak solutions since the system is not strictly hyperbolic.

We find eigenvalues λ_1, λ_2 of the Jacobian of the flux $F = (0, kg(u))$. Clearly, $\lambda_1 = 0$, $\lambda_2 = kg'(u)$. Since both characteristics fields are either linearly degenerate (the first one) or genuinely nonlinear (the second one) on the set $(k_R, k_L) \times (0,1)$ the Lax conditions are rather natural demand here. It is not difficult to see that an admissible weak solution to (3) has the form:

$$k(x,t) = k(x) = \begin{cases} k_L, & x < 0 \\ k_R, & x > 0, \end{cases} \qquad u(x,t) = \begin{cases} u_L, & x < \frac{g(u_L)}{u_L - 1} t, \\ 1, & \frac{g(u_L)}{u_L - 1} t < x < 0, \\ 0, & 0 < x < \frac{g(u_R)}{u_R} t, \\ u_R, & \frac{g(u_R)}{u_R} t < x, \end{cases} \tag{4}$$

where the states (k_L, u_L) and $(k_L, 1)$, as well as $(k_R, 0)$ and (k_R, u_R) are connected by 2-shock waves, while the states $(k_L, 1)$ and $(k_R, 0)$ are connected by 1-contact discontinuity.

So, one of 'natural' jumps with discontinuity at the interface $x_0 = 0$ is the one connecting $u_L = 1$ and $u_R = 0$. Since u_L and u_R from (3) are arbitrary, the latter implies that any admissible weak solution should have values close to $u_L = 1$ in the left neighborhood of $x_0 = 0$ and values close to $u_R = 0$ in the right neighborhood of $x_0 = 0$ which is obvious shortcoming of our admissibility concept. On the other hand, it appears that we can apply similar concept for (1).

Furthermore, if we model e.g. traffic flow problem, solutions such as ours are possible. For instance, if we are on the road which is divided by a point which is completely impermeable (such as a ramp) then we will have exactly the solution which we propose. Furthermore, in our admissibility setting, it is easy to prove uniqueness – the paper perfectly fits into the standard Kruzhkov theory, and also to construct admissible solution to (1).

We stress that, although it is a rather special, the mentioned situation was not covered in any of the previous works.

First, we need auxiliary notion of admissibility. $L^1(I\!R)$-closure of the set of weak solutions to (1) satisfying such admissibility conditions will be set of all admissible weak solutions.

Definition 1. Let $u \in L^\infty(\mathbf{R}\times\mathbf{R})$ represents a weak solution to (1). We say that u is the germ admissible weak solution (g.a.w.s) to (1) if for every $c \in \mathbf{R}$ and every non-negative $\varphi \in C_0^\infty([0,T)\times\mathbf{R})$ there exists a positive constant σ_0 such that for every $\sigma < \sigma_0$, it holds:

$$\begin{aligned}
&\int_0^T \int_0^\infty |u(x,t)-c|\partial_t\varphi(x,t)dx \qquad (5)\\
&+\int_0^T\int_0^\infty \operatorname{sgn}(u(x,t)-c)(f(u(x,t))-f(c))\partial_x\varphi(x,t)dxdt\\
&+\int_0^\infty |u_0(x)-c|\varphi(0,x)dx - \int_0^T \frac{1}{\sigma}\int_0^\sigma (f(u(x,t))-f(c))\varphi(x,t)dxdt \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_{-\infty}^0 |u(x,t)-c|\partial_t\varphi(x,t)dxdt \qquad (6)\\
&+\int_0^T\int_{-\infty}^0 \operatorname{sgn}(u(x,t)-c)(g(u(x,t))-g(c))\partial_x\varphi(x,t)dxdt\\
&+\int_{-\infty}^0 |u_0(x)-c|\varphi(0,x)dx - \int_0^T \frac{1}{\sigma}\int_{-\sigma}^0 (g(u(x,t))-g(c))\varphi(0,t)dt \geq 0.
\end{aligned}$$

Theorem 2. *For every $T>0$ and every non-negative $\varphi \in C_0^\infty([0,T)\times\mathbf{R})$, the g.a.w.s. solutions u and v to (1) with the initial conditions u_0 and v_0 satisfy for almost every $t \in [0,T)$:*

$$\int_{\mathbf{R}} |u(x,t)-v(x,t)|dx \leq \int_{\mathbf{R}} |u_0(x)-v_0(x)|dx. \qquad (7)$$

Proof: We use standard doubling of variables technique. Denote by $\Phi_L(u,v) = \operatorname{sgn}(u-v)(g(u)-g(v))$ and $\Phi_R(u,v) = \operatorname{sgn}(u-v)(f(u)-f(v))$.

Instead of c in (5) we put $c = v(\bar{x},\bar{t})$. Instead of ϕ from (5) we put $\varphi = \varphi(x,t,\bar{x},\bar{t}) = \psi(\frac{x+\bar{x}}{\nu},\frac{t+\bar{t}}{\nu})\frac{1}{\nu^2}\rho(\frac{t-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})$.

We get after integrating (5) over $(\bar{x},\bar{t}) \in (0,\infty)\times[0,T)$:

$$\int_0^T\int_0^\infty\int_0^T\int_0^\infty |u(x,t)-v(\bar{x},\bar{t})|\partial_t\varphi + \Phi_R(u(x,t),v(\bar{x},\bar{t})\partial_x\varphi dxdtd\bar{x}d\bar{t} \tag{8}$$
$$-\int_0^T\int_0^\infty\int_0^T\frac{1}{\sigma}\int_0^\sigma (f(u(x,t))-f(v(\bar{x},\bar{t})))\varphi(x,t,\bar{x},\bar{t})dxdtd\bar{x}d\bar{t}$$
$$+\int_0^T\int_0^\infty\int_0^\infty |u_0(x)-v_0(\bar{x},\bar{t})|\varphi(x,0,\bar{x},\bar{t})dxd\bar{x}d\bar{t} \geq 0.$$

Then, we change places to u and v and sum such obtained inequality with (8). We get:

$$\int_0^T\int_0^\infty\int_0^T\int_0^\infty |u(x,t)-v(\bar{x},\bar{t})|\partial_t\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\times \tag{9}$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxdtd\bar{x}d\bar{t}$$
$$+\int_0^T\int_0^\infty\int_0^T\int_0^\infty \Phi_R(u(x,t),v(\bar{x},\bar{t})\partial_x\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxdtd\bar{x}d\bar{t}$$
$$-\int_0^T\int_0^\infty\int_0^T\frac{1}{\sigma}\int_0^\sigma (f(u(x,t))-f(v(\bar{x},\bar{t})))\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxdtd\bar{x}d\bar{t}$$
$$-\int_0^T\int_0^\infty\int_0^T\frac{1}{\sigma}\int_0^\sigma (f(v(x,t))-f(u(\bar{x},\bar{t})))\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxdtd\bar{x}d\bar{t}$$
$$+\int_0^T\int_0^\infty\int_0^\infty |u_0(x)-v(\bar{x},\bar{t})|\psi(\frac{x+\bar{x}}{2},\frac{\bar{t}}{2})\frac{1}{\nu^2}\rho(\frac{-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxd\bar{x}d\bar{t}$$
$$+\int_0^T\int_0^\infty\int_0^\infty |u(\bar{x},\bar{t})-v_0(x)|\psi(\frac{x+\bar{x}}{2},\frac{\bar{t}}{2})\frac{1}{\nu^2}\rho(\frac{-\bar{t}}{2\nu})\rho(\frac{x-\bar{x}}{2\nu})dxd\bar{x}d\bar{t} \geq 0.$$

We apply the same procedure on inequality (6) and get

$$\int_0^T\int_{-\infty}^0\int_0^T\int_{-\infty}^0 |u(x,t)-v(\bar x,\bar t)|\partial_t\psi(\frac{x+\bar x}{2},\frac{t+\bar t}{2})\times \tag{10}$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxdtd\bar x d\bar t$$
$$+\int_0^T\int_{-\infty}^0\int_0^T\int_{-\infty}^0 \Phi_L(u(x,t),v(\bar x,\bar t))\partial_x\psi(\frac{x+\bar x}{2},\frac{t+\bar t}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxdtd\bar x d\bar t$$
$$-\int_0^T\int_{-\infty}^0\int_0^T\frac{1}{\sigma}\int_{-\sigma}^0(g(u(x,t))-g(v(\bar x,\bar t)))\psi(\frac{x+\bar x}{2},\frac{t+\bar t}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxdtd\bar x d\bar t$$
$$-\int_0^T\int_{-\infty}^0\int_0^T\frac{1}{\sigma}\int_{-\sigma}^0(g(v(x,t))-g(u(\bar x,\bar t)))\psi(\frac{x+\bar x}{2},\frac{t+\bar t}{2})\times$$
$$\times\frac{1}{\nu^2}\rho(\frac{t-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxdtd\bar x d\bar t$$

$$+\int_0^T\int_{-\infty}^0\int_{-\infty}^0 |u_0(x)-v(\bar x,\bar t)|\psi(\frac{x+\bar x}{2},\frac{\bar t}{2})\frac{1}{\nu^2}\rho(\frac{-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxd\bar x d\bar t$$
$$+\int_0^T\int_{-\infty}^0\int_{-\infty}^0 |u(\bar x,\bar t)-v_0(x)|\psi(\frac{x+\bar x}{2},\frac{\bar t}{2})\frac{1}{\nu^2}\rho(\frac{-\bar t}{2\nu})\rho(\frac{x-\bar x}{2\nu})dxd\bar x d\bar t\geq 0.$$

Then, we add (9) and (10) and let $\nu\to 0$. We get:

$$\int_0^T\int_{\mathbf{R}}|u(x,t)-v(x,t)|\partial_t\psi(x,t)dx \tag{11}$$
$$+\int_0^T\int_{\mathbf{R}}\left(\Phi_L(u(x,t),v(x,t))+\Phi_R(u(x,t),v(x,t))\right)\partial_x\psi(x,t)dxdt$$
$$+\int_{\mathbf{R}}|u_0(x)-v_0(x)|\psi(0,x)dx\geq 0.$$

Estimate (7) follows from (11) by standard choice of the test function (see [10]).

□

Notice that the shock wave (more precisely, the contact discontinuity):

$$u(t,x)=\begin{cases}1, & x<0\\ 0, & x\geq 0\end{cases},\quad t\in I\!R^+,$$

is a g.a.w.s. to the conservation law from (1) with appropriate initial conditions. Also, notice that any g.a.w.s. to (1) must satisfy the Kruzhkov entropy admissible conditions for test functions supported out of the interface $x_0 = 0$.

Using this observation, we can prove:

Theorem 3. *For every function* $u_0 \in L^1(I\!R^d)$, $0 \leq u_0 \leq 1$, *such that there exists* $\varepsilon > 0$ *satisfying*

$$u_0(x) = \begin{cases} 1, & -\varepsilon \leq x < 0 \\ 0, & 0 \leq x < \varepsilon \end{cases} \tag{12}$$

there exists g.a.w.s. to (1).

Proof: First, denote by

$$u_0^L(x) = \begin{cases} u_0(x), & x \leq -\varepsilon \\ 1, & x > -\varepsilon. \end{cases}$$

Then, denote by $u^L \in L^1 \cap L^\infty(I\!R \times I\!R^+)$ the Kruzhkov entropy admissible solution to the following Cauchy problem:

$$\partial_t u^L + \partial_x g(u^L) = 0$$
$$u|_{t=0} = u_0^L.$$

Clearly, it holds:

$$u^L(x,t) \equiv 1, \quad t \in I\!R^+, \quad x \leq -\varepsilon,$$

and, according to the maximum principle, $0 \leq u^L \leq 1$.

Indeed, notice that if the left state of a weak solution u to (1) is equal to one then only admissible simple wave corresponding to such left state is either (see Figure 1)

- the shock wave moving toward $-\infty$ since it moves with the velocity $c = \frac{g(1)-g(U)}{1-U} \leq 0$ (since $0 \leq U \leq 1$), and the germ entropy admissibility conditions coincides with the standard Kruzhkov entropy admissibility conditions for $x \leq -\varepsilon$; or
- the rarefaction wave whose points moves along the characteristics. Since $g'(1) \leq 0$, a point corresponding to the state $u = 1$ must move away from the interface $x_0 = 0$.

Similarly, let $u^R \in L^1 \cap L^\infty(I\!R \times I\!R^+)$ the Kruzhkov entropy admissible solution to the following Cauchy problem:

$$\partial_t u^R + \partial_x g(u^R) = 0$$
$$u^R|_{t=0} = \begin{cases} u_0(x), & x \geq \varepsilon \\ 0, & x < \varepsilon. \end{cases}.$$

As for the function u^L, it holds:

$$u^R(x,t) \equiv 1, \quad t \in I\!R^+, \quad x \geq \varepsilon,$$

and $0 \leq u^R \leq 1$.

The germ weak admissible weak solution to (1) where u_0 is given by (12) is given by:

$$u(x,t) = \begin{cases} u^L(x,t), & x \leq -\varepsilon \\ 1, & -\varepsilon \leq x < 0 \\ 0, & 0 \leq x < \varepsilon \\ u^R(x,t), & x \geq \varepsilon, \end{cases}$$

since in a neighborhood of the interface $x_0 = 0$ we have the germ weak admissible contact discontinuity, while out of the neighborhood the Kruzhkov admissibility conditions are satisfied.

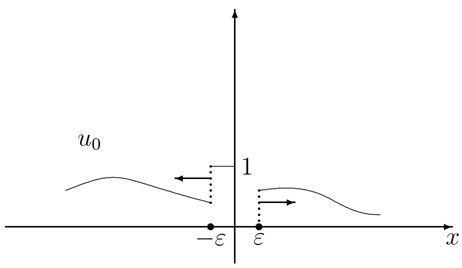

Figure 1: Arrows show directions of propagation of appropriate waves.

□

Now, we introduce definition of thickened admissibility. The term "thickened" was (indirectly) proposed by J.Vovelle.

Definition 4. We say that the function $u \in L^1 \cap L^\infty(I\!R^+ \times I\!R)$ is a thickened admissible weak solution (t.a.w.s.) to (1) if for every $\varepsilon > 0$ there exists a g.a.w.s. $u_\varepsilon \in L^1 \cap L^\infty(I\!R)$ corresponding to the initial conditions $u_{0\varepsilon} \in L^1 \cap L^\infty(I\!R)$ such that $\|u - u_\varepsilon\|_{L^1(I\!R\times[0,T])} \leq T\varepsilon$, $T > 0$, and $\|u_0 - u_{0\varepsilon}\|_{L^1(I\!R)} \leq \varepsilon$.

Theorem 5. *For every $u_0 \in L^1 \cap L^\infty(I\!R)$, there exists a unique t.a.w.s. to* (1).

Moreover, any two a.w.s. u and v to (1) *corresponding to the initial data u_0 and v_0, respectively, satisfy:*

$$\|u - v\|_{L^1([0,T]\times I\!R)} \leq T\|u_0 - v_0\|_{L^1(I\!R)}. \tag{13}$$

Proof: First, we construct the g.a.w.s. u_ε such that $\|u-u_\varepsilon\|_{L^1(I\!R\times[0,T])} \leq T\varepsilon$.

We consider the following approximation to (1):

$$\begin{cases} \partial_t u + \partial_x(g(u)H(-x) + f(u)H(x)) = 0, \\ u|_{t=0} = u_{0\varepsilon}(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}), \end{cases} \tag{14}$$

where

$$u_{0\varepsilon}(x) = \begin{cases} 1, & -\varepsilon \leq x < 0 \\ 0, & 0 \leq x < \varepsilon \\ u_0(x), & else. \end{cases}$$

According to Theorem 3, there exists g.a.w.s. u_ε to (14).

Now, for two parameters $\varepsilon_1, \varepsilon_2 > 0$, we have two germ admissible solutions u_{ε_1} and u_{ε_2} to (14) with the initial conditions $u_{0\varepsilon_1}$ and $u_{0\varepsilon_2}$. According to Theorem 2, it holds

$$\int_{\mathbf{R}} |u_{\varepsilon_1}(x,t) - u_{\varepsilon_2}(x,t)|dx \leq \int_{\mathbf{R}} |u_{0\varepsilon_1}(x) - u_{0\varepsilon_2}(x)|dx,$$

implying that u_ε strongly converges in $L^1_{loc}(I\!R \times I\!R^+)$.

Clearly, L^1_{loc} limit of the family (u_ε) will represent the thickened admissible weak solution to (1) in the sense of Definition 4.

Now, take two t.a.w.s. u and v to (1) corresponding to the initial data u_0 and v_0, respectively. It holds for appropriate g.w.a.s. u_ε and v_ε according to Theorem 2:

$$\begin{aligned} &\|u-v\|_{L^1(I\!R\times[0,T])} \\ &\leq \|u-u_\varepsilon\|_{L^1(I\!R\times[0,T])} + \|u_\varepsilon - v_\varepsilon\|_{L^1(I\!R\times[0,T])} + \|v-v_\varepsilon\|_{L^1(I\!R\times[0,T])} \\ &\leq \varepsilon + T\|u_{0\varepsilon} - v_{0\varepsilon}\|_{L^1(I\!R^d)} + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ here, we immediately reach to (13).
□

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