Mathematica Aeterna, Vol. 7, 2017, no. 4, 455 - 462

Right distributive QI-algebras with pseudo-valuations

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Abstract

Using the Buşneag's model, the notion of a (pseudo-)valuation on a right distributive QI-algebra and a (pseudo-)metric induced by a (pseudo-)valuation is given. It is shown, that a binary operation in right distributive QI-algebra is uniformly continuous.

Mathematics Subject Classification: 03G25, 06F05, 20N02

Keywords: (right distributive) QI-algebra, (pseudo-) valuation, (pseudo-) metric space

1 Introduction

Buşneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo-metric on a Hilbert algebra.The notion of QI-algebras were introduced by Bandaru in [1].

In this paper, using the Buşneag's model, we introduce the notion of pseudo-valuations (valuations) on right distributive QI-algebras and we give a notion of a pseudo-metric by using a pseudo-valuation on a right distributive QI-algebra. We show that teh binary operation $*$ in right distributive QI-algebras such that, if $x*y = y*x = 0$, then $x = y$, is uniformly continuous.

2 Preliminaries

Definition 2.1 ([1]) *A QI-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:*

(QI1) $x * x = 0$,

(QI2) $x * 0 = x$,

(QI3) $x * (y * (x * y)) = x * y$,

for all $x, y \in X$.

In [1], there was defined a binary relation $\leq$ on X by

$$x \leq y \Leftrightarrow x * y = 0,$$

for all $x, y \in X$.

Example 2.2 ([1]) *Let* $X = \{0, a, b, c\}$ *be a set and let us define an operation* $*$ *on* X *by the table below:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	b
c	c	0	a	0

.

Then $(X, *, 0)$ is a QI-algebra.

Definition 2.3 ([1]) *Let* X *be a QI-algebra. A set* $I \subseteq X$ *is called an ideal, if it satisfies the following conditions for all* $x, y \in X$:

(I1) $0 \in I$;

(I2) If $x * y \in I$ *and* $y \in I$, *then* $x \in I$.

Definition 2.4 ([1]) *A QI-algebra* X *is said to be right distributive if*

(QI4) $(x * y) * z = (x * z) * (y * z)$.

Example 2.5 ([1]) *A QI-algebra from Example 2.2 is a right distributive QI-algebra.*

Proposition 2.6 ([1]) *Let* $(X, *, 0)$ *be a right distributive QI-algebra. Then*

(i) if $x \leq 0$, *then* $x = 0$;

(ii) $(x * y) * y) \leq x$;

(iii) $(x * y) * (x * z) \leq z * y$;

(iv) $(x * z) * (y * z) \leq x * y$.

Definition 2.7 *Let X be a right distributive QI-algebra and $S \subseteq X$. A set S is called a subalgebra of X if S is a right distributive QI-algebra with the same operation as X.*

Proposition 2.8 *Let X be a right distributive QI-algebra and $S \subseteq X$. A set S is a subalgebra of X if and only if S satisfies the following conditions for all $x, y \in X$*

(SQ1) $0 \in S$;

(SQ2) *if $x, y \in S$, then $x * y \in S$.*

3 Pseudo-valuations on a right distributive QI-algebras

Definition 3.1 *A real-valued function φ on a right distributive QI-algebra X is called a weak pseudo-valuation on X if it satisfies the following condition for all $x, y \in X$:*

$$\varphi(x * y) \leq \varphi(x) + \varphi(y).$$

Example 3.2 *Let $X = \{0, a, b, c\}$ be a right distributive QI-algebra from Example 2.2. Let φ be a real-valued function on X defined by*

$$\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 1 & 2 \end{pmatrix}.$$

Then φ is a weak pseudo-valuation on X.

Proposition 3.3 *For a weak pseudo-valuation φ on a right distributive QI-algebra X, we have*

$$\varphi(x) \geq 0$$

for any $x \in X$.

Proof. *Since for any $x \in X$, we have $\varphi(0) = \varphi(0 * x) \leq \varphi(0) + \varphi(x)$, then $\varphi(x) \geq 0$.* ■

Let S be a subalgebra of a right distributive QI-algebra X and let $a, b \in \mathbb{R}^+$ with $a < b$. Let us define a real-valued function φ_S as follows:

$$\varphi_S(x) = \begin{cases} a \text{ if } x \in S \\ b \text{ if } x \notin S \end{cases}.$$

Theorem 3.4 *A function φ_S is a weak pseudo-valuation on a right distributive QI-algebra X.*

Proof. *Straightforward.* ■

Definition 3.5 *A real-valued function φ on a right distributive QI-algebra X is called a pseudo-valuation on X, if it satisfies two following conditions for all $x, y \in X$:*

(PV1) $\varphi(0) = 0$;

(PV2) $\varphi(x * y) \geq \varphi(x) - \varphi(y)$.

Definition 3.6 *Let φ be a pseudo-valuation on a a right distributive QI-algebra X. A function φ is called a valuation, if it satisfies the following condition for any $x \in X$:*

(V) $x \neq 0 \Rightarrow \varphi(x) \neq 0$.

Example 3.7 *Let $X = \{0, a, b, c\}$ be a right distributive QI-algebra from Example 2.2. Let φ be a real-valued function on X defined by*

$$\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 1 & 3 \end{pmatrix}.$$

Then φ is a pseudo-valuation on X. What is more, it is a valuation on X.

Example 3.8 *For any ideal I of a right distributive QI-algebra X we define a real-valued function φ_I on X as follows:*

$$\varphi_I(x) = \begin{cases} 0 \text{ if } x = 0 \\ a \text{ if } x \in I - \{0\} \\ b \text{ if } x \in X - I \end{cases},$$

for all $x \in X$ where $0 < a < b$. A function φ_I is a pseudo-valuation on X which is a valuation on X.

Theorem 3.9 *In a right distributive QI-algebra, every pseudo-valuation is a weak pseudo-valuation.*
Proof. *Let X be a right distributive QI-algebra and $x, y \in X$. From Proposition 2.6 (ii), we have $(x * y) * y) * x = 0$. Hence for a pseudo-valuation φ, $0 = \varphi(0) = \varphi((x * y) * y) * x) \geq \varphi\left((x * y) * y)\right) - \varphi(x) \geq \varphi\left(x * y\right) - \varphi(y) - \varphi(x)$. We get that $\varphi(x * y) \leq \varphi(x) + \varphi(y)$ and therefore φ is a weak pseudo-valuation on X.* ■

Remark 3.10 *The converse of Theorem 3.9 is not true, in general. A weak pseudo-valuation from Example 3.2 is not a pseudo-valuation on X since $\varphi(a) - \varphi(c) = 3 - 2 = 1$ and $\varphi(a * c) = 0$.*

Proposition 3.11 *Let φ be a pseudo-valuation on a right distributive QI-algebra X. Then for all $x, y \in X$:*

(i) φ *is order preserving;*

(ii) $\varphi(x * y) + \varphi(y * x) \geq 0$;

(iii) $\varphi(x * y) \leq \varphi(x * z) + \varphi(z * y)$.

Proof. *(i) Let* $x, y \in X$ *be such that* $x \leq y$ *and* φ *be a pseudo-valuation on* X*. Then we have* $\varphi(x) \leq \varphi(x * y) + \varphi(y)$*. Since* $x \leq y$*, then* $\varphi(x * y) = \varphi(0) = 0$*. Finally, we get* $\varphi(x) \leq \varphi(y)$*.*

(ii) For any pseudo-valuation φ *we have* $\varphi(x*y)+\varphi(y*x) \geq \varphi(x)-\varphi(y)+\varphi(y) - \varphi(x) = 0$*.*

(iii) Let φ *be a pseudo-valuation on* X*. Using Proposition 2.6 and (i) we get* $\varphi(z * y) \geq \varphi\left((x * y) * (x * z)\right) \geq \varphi(x * y) - \varphi(x * z)$*. Hence* $\varphi(x * y) \leq \varphi(x * z) + \varphi(z * y)$*.* ■

Theorem 3.12 *Let* φ *be a real-valued function on a right distributive QI-algebra* X*. If* φ *satisfies (PV1) and for all* $x, y, z \in X$

(PV5) $\varphi(((x * y) * y) * z) \geq \varphi(x * y) - \varphi(z)$,

then φ *is a pseudo-valuation on* X*.*

Proof. *Taking* $y = 0$ *and using (QI2), we get* $\varphi(x * y) = \varphi(((x * 0) * 0) * y) \geq \varphi(x * 0) - \varphi(y) = \varphi(x) - \varphi(y)$*.* ■

Theorem 3.13 *Let* X *be a right distributive QI-algebra and let* φ *be a pseudo-valuation on* X*. Then a set*

$$I = \{x \in X : \varphi(x) = 0\}$$

is an ideal of X*.*

Proof. *Since* φ *is a pseudo-valuation, then* $\varphi(0) = 0$*. Hence* $0 \in I$*. Now, let* $x*y, y \in I$*. Then* $\varphi(x*y) = \varphi(y) = 0$*. Using (PV2) we obtain* $0 \geq \varphi(x) - 0$*.By Theorem 3.9 and Proposition 3.3, we get* $\varphi(x) = 0$*. Hence* $x \in I$*.* ■

For a real-valued function on a right distributive QI-algebra X, define a mapping $d_\varphi : X \times X \to \mathbb{R}$ by $d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x)$ for all $x, y \in X$.

Definition 3.14 *A positive function* $d : X \times X \to \mathbb{R}$ *is called a pseudo-metric on* X*, if it satisfies the following conditions for all* $x, y \in X$ *:*

(PM1) $d(x, x) = 0$;

(PM2) $d(x, y) = d(y, x)$;

(PM3) $d(x, z) \leq d(x, y) + d(y, z)$.

Theorem 3.15 *Let X be a right distributive QI-algebra and let φ be a pseudo-valuation on X. Then d_φ is a pseudo-metric on X and so (X, d_φ) is a pseudo-metric space.*

Proof. *It is obvious that $d_\varphi(x,y) \geq 0$. Since $x * x = 0$, then $d_\varphi(x,x) = \varphi(x,x) + \varphi(x,x) = 0 + 0 = 0$. Obviously $d_\varphi(x,y) = d_\varphi(y,x)$. We need to show, that $d_\varphi(x,z) \leq d_\varphi(x,y) + d_\varphi(y,z)$. By Proposition 3.11, we have*

$$\begin{aligned} d_\varphi(x,y) + d_\varphi(y,z) &= (\varphi(x*y) + \varphi(y*x)) + (\varphi(y*z) + \varphi(z*y)) = \\ &= (\varphi(x*y) + \varphi(y*z)) + (\varphi(z*y) + \varphi(y,x)) \geq \\ &\geq \varphi(x*z) + \varphi(z*x) = d_\varphi(x,z). \end{aligned}$$

Therefore (X, d_φ) is a pseudo-metric space. ■

We say d_φ is a pseudo-metric induced by a pseudo-valuation φ.

Proposition 3.16 *Let X be a right distributive QI-algebra. Then for every pseudo-metric d_φ induced by a pseudo-valuation φ we have:*

(i) $d_\varphi(x,y) \geq d_\varphi(x*a, y*a)$;

(ii) $d_\varphi(x,y) \geq d_\varphi(a*x, a*y)$;

(iii) $d_\varphi(x*y, a*b) \leq d_\varphi(x*y, a*y) + d_\varphi(a*y, a*b)$,
for all $x, y, a, b \in X$.

Proof. *(i) Using Proposition 2.6 (iv) and Proposition 3.11 (i) we get $\varphi((x*a)*(y*a)) \leq \varphi(x*y)$ and $\varphi((y*a)*(x*a)) \leq \varphi(y*x)$ for all $x, y, a \in X$. Hence*

$$\begin{aligned} d_\varphi(x,y) &= \varphi(x*y) + \varphi(y*x) \geq \\ &\geq \varphi((x*a)*(y*a)) + \varphi((y*a)*(x*a)) = \\ &= d_\varphi(x*a, y*a). \end{aligned}$$

(ii) Similarly to (i).

(iii) Using Proposition 3.11 (iii)

$$\varphi((x*y)*(a*b)) \leq \varphi((x*y)*(a*y)) + \varphi((a*y)*(a*b))$$

and

$$\varphi((a*b)*(x*y)) \leq \varphi((a*b)*(a*y)) + \varphi((a*y)*(x*y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_\varphi(x*y,a*b) &= \varphi((x*y)*(a*b))+\varphi((a*b)*(x*y)) \leq \\ &\leq \varphi((x*y)*(a*y))+\varphi((a*y)*(a*b))+ \\ &+\varphi((a*b)*(a*y))+\varphi((a*y)*(x*y)) = \\ &= [\varphi((x*y)*(a*y))+\varphi((a*y)*(x*y))]+ \\ &+[\varphi((a*y)*(a*b))+\varphi((a*b)*(a*y))] = \\ &= d_\varphi(x*y,a*y)+d_\varphi(a*y,a*b) \end{aligned}$$

for all $x,y,a,b\in X$. ■

Theorem 3.17 *Let* φ *be a pseudo-valuation on a right distributive QI-algebra. Then* $\left(X\times X,d_\varphi^*\right)$, *where*

$$d_\varphi^*((x,y),(a,b)) = \max\left(d_\varphi(x,a),d_\varphi(y,b)\right),$$

for all $(x,y),(a,b)\in X\times X$, *is a pseudo-metric space.*
Proof. *Let* φ *be a pseudo-valuation on* X. *Then*

$$d_\varphi^*((x,y),(x,y)) = \max\left(d_\varphi(x,x),d_\varphi(y,y)\right) = 0.$$

It is obvious that $d_\varphi^*((x,y),(a,b)) = d_\varphi^*((a,b),(x,y))$. *Now, let* $(x,y),(a,b),(c,d)\in X\times X$. *We get that*

$$\begin{aligned} d_\varphi^*((x,y),(c,d))+d_\varphi^*((c,d),(a,b)) &= \max\left(d_\varphi(x,c),d_\varphi(y,d)\right)+ \\ &+\max\left(d_\varphi(c,a),d_\varphi(d,b)\right)\geq \\ &\geq \max\left(d_\varphi(x,c)+d_\varphi(c,a),d_\varphi(y,d)+d_\varphi(d,b)\right) \\ &\geq \max\left(d_\varphi(x,a),d_\varphi(y,b)\right) \\ &= d_\varphi^*((x,y),(a,b)). \end{aligned}$$

Therefore $\left(X\times X,d_\varphi^*\right)$ *is a pseudo-metric space.* ■

Example 3.18 *Taking a valuation from Example 3.7 we obtain*

$$d_\varphi = \begin{pmatrix} (0,0) & (0,a) & (0,b) & (0,c) & (a,a) & (a,b) & (a,c) & (b,b) & (b,c) & (c,c) \\ 0 & 3 & 1 & 3 & 0 & 3 & 0 & 0 & 3 & 0 \end{pmatrix}.$$

Where (X,d_φ) *is a pseudo-metric space.*

Remark 3.19 d_φ *is not a metric, in general. Let us consider a pseudo-metric from Example 3.18. Then we have* $d_\varphi(a,c)=0$ *and obviously* $a\neq c$.

Theorem 3.20 *Let X be a right-distributive QI-algebra such that*

$$(\forall x, y \in X)\,((x * y = 0 \wedge y * x = 0) \Rightarrow x = y) \tag{1}$$

and let φ be a valuation on X. Then (X, d_φ) is a metric space.

Proof. *Let φ be a valuation on X.Then by Theorem 3.15 d_φ is a pseudo-metric. We need to show, that if $d_\varphi(x, y) = 0$, then $x = y$. Let $x, y \in X$ be such that $d_\varphi(x, y) = 0$. Then $\varphi(x * y) + \varphi(y * x) = 0$. Using Proposition 3.3 and condition (1), we get that $\varphi(x * y) = 0$ and $\varphi(y * x) = 0$. Hence $x * y = 0$ and $y * x = 0$ and so $x = y$. Therefore $d_\varphi$is a metric.* ■

Theorem 3.21 *Let φ be a valuation on a right distributive QI-algebra X satisfying a condition (1). Then $(X \times X, d^*_\varphi)$ is a metric space.*

Proof. *Let φ be a valuation on X. Then by Theorem 3.17 $(X \times X, d^*_\varphi)$ is a pseudo-metric space. Now, suppose $(x, y), (a, b) \in X \times X$ are such that $d^*_\varphi((x, y), (a, b)) = 0$. Hence $\max(d_\varphi(x, a), d_\varphi(y, b)) = 0$. By Definition of $d^*_\varphi$and Proposition 3.3, $d_\varphi(x, a) = d_\varphi(y, b) = 0$. Therefore $x = a$ and $y = b$. Hence $(x, y) = (a, b)$ and so $(X \times X, d^*_\varphi)$ is a metric space.* ■

Theorem 3.22 *Let φ be a valuation on a right distributive QI-algebra satisfying a condition (1). The operation $*$ is uniformly continuous.*

Proof. *Let $\varepsilon > 0$ and suppose that $d^*_\varphi((x, y), (a, b)) < \frac{\varepsilon}{2}$.Then $d_\varphi(x, a) < \frac{\varepsilon}{2}$ and $d_\varphi(y, b) < \frac{\varepsilon}{2}$. Using Proposition 3.16, we obtain*

$$d_\varphi(x * y, a * b) \leq d_\varphi(x * y, a * y) + d_\varphi(a * y, a * b) \leq d_\varphi(x, a) + d_\varphi(y, b) < \varepsilon.$$

Therefore $* : X \times X \to X$ is uniformly continuous. ■

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Received: December 29, 2017