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Radial Signed Digraphs

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Abstract

In this paper, we define the radial signed digraph $\overrightarrow{R}(\Sigma)$ of a given signed digraph $\Sigma = (\Delta, \sigma)$ and offer a structural characterization of radial signed digraphs. Further, we characterize signed digraphs Σ for which: $\Sigma \sim \overrightarrow{R}(\Sigma)$ and $\overline{\Sigma} \sim \overrightarrow{R}(\Sigma)$ where $\sim$ denotes switching equivalence and $\overrightarrow{R}(\Sigma)$ and $\overline{\Sigma}$ are denotes the radial signed digraph and complementary signed digraph of Σ respectively.

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1 Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic text-books of Bondy and Murty [1] and Harary et al. [2]; the non-standard will be given in this paper as and when required.

A *signed digraph* is an ordered pair $\Sigma = (\Delta, \sigma)$, where $\Delta = (V, \mathcal{A})$ is a digraph called *underlying digraph of* Σ and $\sigma : \mathcal{A} \rightarrow \{+, -\}$ is a function. A *marking* of Σ is a function $\zeta : V(\Delta) \rightarrow \{+, -\}$. A signed digraph Σ together with a marking ζ is denoted by Σ_ζ. A signed digraph $\Sigma = (\Delta, \sigma)$ is *balanced* if every semicycle of Σ is positive (See [2]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained in [4].

Theorem 1.1. (**E. Sampathkumar et al.** [**4**]) *A signed digraph* $\Sigma = (\Delta, \sigma)$ *is balanced if, and only if, there exist a marking* ζ *of its vertices such that each arc* $\overrightarrow{uv}$ *in* Σ *satisfies* $\sigma(\overrightarrow{uv}) = \zeta(u)\zeta(v)$.

Let $\Sigma = (\Delta, \sigma)$ be a signed digraph. Consider the marking ζ on vertices of Σ defined as follows: each vertex $v \in V$, $\zeta(v)$ is the product of the signs on the arcs incident at v. *Complement* of Σ is a signed digraph $\overline{\Sigma} = (\overline{\Delta}, \sigma')$, where for any arc $e = \overrightarrow{uv} \in \overline{\Delta}$, $\sigma'(\overrightarrow{uv}) = \zeta(u)\zeta(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed digraph due to Theorem 1.1.

In [4], the authors define switching and cycle isomorphism of a signed digraph as follows:

Let $\Sigma = (\Delta, \sigma)$ and $\Sigma' = (\Delta', \sigma')$, be two signed digraphs. Then Σ and Σ' are said to be *isomorphic*, if there exists an isomorphism $\phi : \Delta \rightarrow \Delta'$ (that is a bijection $\phi : V(\Delta) \rightarrow V(\Delta')$ such that if $\overrightarrow{uv}$ is an arc in Δ then $\overrightarrow{\phi(u)\phi(v)}$ is an arc in Δ') such that for any arc $\overrightarrow{e} \in \Delta$, $\sigma(\overrightarrow{e}) = \sigma'(\phi(\overrightarrow{e}))$.

Given a marking ζ of a signed digraph $\Sigma = (\Delta, \sigma)$, *switching* Σ with respect to ζ is the operation changing the sign of every arc $\overrightarrow{uv}$ of Σ' by $\zeta(u)\sigma(\overrightarrow{uv})\zeta(v)$. The signed digraph obtained in this way is denoted by $\sum_\zeta(\Sigma)$ and is called ζ *switched signed digraph* or just *switched signed digraph*.

Further, a signed digraph Σ switches to signed digraph Σ' (or that they are switching equivalent to each other), written as $\Sigma \sim \Sigma'$, whenever there exists a marking of Σ such that $\sum_\zeta(\Sigma) \cong \Sigma'$.

Two signed digraphs $\Sigma = (\Delta, \sigma)$ and $\Sigma' = (\Delta', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : \Delta \rightarrow \Delta'$ such that the sign $\sigma(Z)$ of every semicycle Z in Σ equals to the sign $\sigma(\phi(Z))$ in Σ'.

Theorem 1.2. (**E. Sampathkumar et al.** [**4**])
Two signed digraphs Σ_1 *and* Σ_2 *with the same underlying digraph are switching equivalent if, and only if, they are cycle isomorphic.*

2 Radial Signed Digraph of a Signed Digraph

For a pair u, v of vertices in a strong digraph Δ the distance $d(u;v)$ is the length of a shortest directed $u-v$ path. The eccentricity of a vertex u, denoted by $e(u)$, is the maximum distance from u to any vertex in Δ. The radius of Δ, $\text{rad}(\Delta)$, is the minimum eccentricity of the vertices in Δ; the diameter, $\text{diam}(\Delta)$, is the maximum eccentricity of the vertices in Δ. Let $S_i(\Delta)$ be the subset of the vertex set of Δ consisting of vertices with eccentricity i.

Kathiresan and Sumathi [3] introduced a new type of digraph called radial digraph. For a digraph Δ, the Radial digraph $\overrightarrow{R}(\Delta)$ of Δ is the digraph with $V(\overrightarrow{R}(\Delta)) = V(\Delta)$, and $E(\overrightarrow{R}(\Delta)) = \{(u,v); u, v \in V(\Delta)$ and $d_\Delta(u;v) = rad(\Delta)\}$. A digraph Δ is called a radial digraph if $\overrightarrow{R}(\Delta') = \Delta$ for some digraph Δ'.

We extend the notion of $\overrightarrow{R}(D)$ to the realm of signed digraphs. In a signed digraph $\Sigma = (\Delta, \sigma)$, where $\Delta = (V, \mathcal{A})$ is a digraph called *underlying digraph of S* and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. The *radial signed digraph* $\overrightarrow{R}(\Sigma) = (\overrightarrow{R}(\Delta), \sigma')$ of a signed digraph $\Sigma = (\Delta, \sigma)$ is a signed digraph whose underlying digraph is $\overrightarrow{R}(D)$ called *radial digraph* and sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{R}(\Sigma)$, $\sigma'(e) = \zeta(u)\zeta(v)$, where for any $v \in V$, $\zeta(v) = \prod_{u \in N(v)} \sigma(uv)$. Further, a signed digraph $\Sigma = (\Delta, \sigma)$ is called *radial signed digraph*, if $\Sigma \cong \overrightarrow{R}(\Sigma')$, for some signed digraph Σ'.

3 Switching Invariant Radial Signed Digraphs

The following result indicates the limitations of the notion $\overrightarrow{R}(\Sigma)$ as introduced above, since the entire class of unbalanced signed digraphs is forbidden to be radial signed digraphs.

Theorem 3.1. *For any signed digraph $S = (D, \sigma)$, its radial signed digraph $\overrightarrow{R}(\Sigma)$ is balanced.*

Proof. Since sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{R}(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ, by Theorem 1.1, $\overrightarrow{R}(\Sigma)$ is balanced. □

For any positive integer k, the k^{th} iterated radial signed digraph $\overrightarrow{R}(\Sigma)$ of Σ is defined as follows:

$$\overrightarrow{R}^0(\Sigma) = \Sigma,\ \overrightarrow{R}^k(\Sigma) = \overrightarrow{R}(\overrightarrow{R}^{k-1}(\Sigma))$$

Corollary 3.2. *For any signed digraph $S = (D, \sigma)$ and any positive integer k, $\overrightarrow{R}^k(S)$ is balanced.*

A digraph Δ is radial invariant, if $\overrightarrow{R}(\Delta) \cong \Delta$. One of the results of Kathiresan and Sumathi [3] characterizing radial invariant graphs is stated below since the main result of this section will need its flavor.

Theorem 3.3. *Let Δ be a digraph of order p. Then $\overrightarrow{R}(\Delta) = \Delta$ if, and only if, $rad(\Delta) = 1$.*

We now characterize the signed digraphs that are switching equivalent to their radial signed digraphs.

Theorem 3.4. *For any signed digraph $\Sigma = (\Delta, \sigma)$, $\Sigma \sim \overrightarrow{R}(\Sigma)$ if, and only if, Σ is balanced signed digraph and $rad(\Delta) = 1$.*

Proof. Suppose $\Sigma \sim \overrightarrow{R}(\Sigma)$. This implies, $\Delta \cong \overrightarrow{R}(\Delta)$ and hence by Theorem 3.3, we have $\text{rad}(\Delta) = 1$. Now, if Σ is any signed digraph with $\text{rad}(\Delta) = 1$, Theorem 3.1 implies that $\overrightarrow{R}(\Sigma)$ is balanced and hence if Σ is unbalanced its $\overrightarrow{R}(\Sigma)$ being balanced cannot be switching equivalent to Σ in accordance with Theorem 1.2. Therefore, Σ must be balanced.

Supposed Σ is balanced signed digraph with $\text{rad}(\Delta) = 1$. Then, since $\overrightarrow{R}(\Sigma)$ is balanced as per Theorem 3.1 and since $\overrightarrow{R}(\Delta) \cong \Delta$ by Theorem 3.3, the result follows from Theorem 1.2 again. □

In [3], the authors characterize the graphs for which $\overrightarrow{R}(\Delta) \cong \overline{\Delta}$.

Theorem 3.5. *Let Δ be a digraph of order p. Then $\overrightarrow{R}(\Delta) \cong \overline{\Delta}$ if, and only if, any one of the following holds:*

i. $S_2(\Delta) = V(\Delta)$

ii. *Δ is not strongly connected such that for any $v \in V(\Delta)$, out degree of $v < p-1$ and for every pair u, v of vertices of Δ, the distance $d_\Delta(u, v) = 1$ or $d_\Delta(u, v) = \infty$.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\overrightarrow{R}(\Sigma) \sim \overline{\Sigma}$.

Theorem 3.6. *For any signed digraph $\Sigma = (\Delta, \sigma)$, $\overrightarrow{R}(\Sigma) \sim \overline{\Sigma}$ if, and only if, Δ satisfies the any one of the conditions of Theorem 3.5.*

Proof. Suppose that $\overrightarrow{R}(\Sigma) \sim \overline{\Sigma}$. Then clearly, $\overrightarrow{R}(\Delta) \cong \overline{\Delta}$. Hence by Theorem 3.5, Δ satisfies the any one of the conditions:

i. $S_2(\Delta) = V(\Delta)$

ii. Δ is not strongly connected such that for any $v \in V(\Delta)$, out degree of $v < p-1$ and for every pair u, v of vertices of Δ, the distance $d_\Delta(u,v) = 1$ or $d_\Delta(u,v) = \infty$.

Conversely, suppose that Σ is a signed digraph whose underlying digraph satisfies the any one of the conditions of Theorem 3.5. Then by Theorem 3.5, $\overrightarrow{R}(\Delta) \cong \overline{\Delta}$. Since for any signed digraph Σ, both $\overrightarrow{R}(\Sigma)$ and $\overline{\Sigma}$ are balanced, the result follows by Theorem 1.2. □

The notion of *negation* $\eta(\Sigma)$ of a given signed digraph Σ defined in [5, 6] as follows: $\eta(\Sigma)$ has the same underlying digraph as that of Σ with the sign of each arc opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

For a signed digraph $\Sigma = (\Delta, \sigma)$, the $\overrightarrow{R}(\Sigma)$ is balanced (Theorem 3.1). We now examine, the conditions under which negation $\eta(\Sigma)$ of $\overrightarrow{R}(\Sigma)$ is balanced.

Theorem 3.7. *Let $\Sigma = (\Delta, \sigma)$ be a signed digraph. If $\overrightarrow{R}(\Delta)$ is bipartite then $\eta(\overrightarrow{R}(\Sigma))$ is balanced.*

Proof. Since, by Theorem 3.1, $\overrightarrow{R}(\Sigma)$ is balanced, if each semicycle C in $\overrightarrow{R}(\Sigma)$ contains even number of negative arcs. Also, since $\overrightarrow{R}(\Delta)$ is bipartite, all semicycles have even length; thus, the number of positive arcs on any semicycle C in $\overrightarrow{R}(\Delta)$ is also even. Hence $\eta(\overrightarrow{R}(\Sigma))$ is balanced. □

3.1 Characterization of Radial Signed Digraphs

In [7], the authors introduced the notion radial signed graph and obtained its characterization. Also, they obtained some switching equivalence characterizations. The following result characterize signed digraphs which are radial signed digraphs.

Theorem 3.8. *A signed digraph $\Sigma = (\Delta, \sigma)$ is a radial signed digraph if, and only if, Σ is balanced signed digraph and its underlying digraph Δ is a radial graph.*

Proof. Suppose that Σ is balanced and Δ is a $\overrightarrow{R}(\Delta)$. Then there exists a digraph Δ' such that $\overrightarrow{R}(\Delta') \cong \Delta$. Since Σ is balanced, by Theorem 1.1, there exists a marking ζ of Δ such that each arc $\overrightarrow{uv}$ in Σ satisfies $\sigma(\overrightarrow{uv}) = \zeta(u)\zeta(v)$. Now consider the signed digraph $\Sigma' = (\Delta', \sigma')$, where for any arc e in Δ', $\sigma'(e)$ is the marking of the corresponding vertex in Δ. Then clearly, $\overrightarrow{R}(\Sigma') \cong \Sigma$.

Hence Σ is a radial signed digraph.

Conversely, suppose that $\Sigma = (\Delta, \sigma)$ is a radial signed digraph. Then there exists a signed digraph $\Sigma' = (\Delta', \sigma')$ such that $\overrightarrow{R}(\Sigma') \cong \Sigma$. Hence Δ is the $\overrightarrow{R}(\Delta)$ of Δ' and by Theorem 3.1, Σ is balanced. $\square$

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