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Oscillation of Fractional Nonlinear Difference Equations

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Abstract

The oscillation criteria for forced nonlinear fractional difference equation of the form

$$\begin{aligned}\Delta^{\alpha}x(t)+f_1(t,x(t+\alpha)) =& v(t)+f_2(t,x(t+\alpha)), \quad t\in N_0, \quad 0<\alpha\leq 1,\\ \Delta^{\alpha-1}x(t)|_{t=0} =& x_0,\end{aligned}$$

where Δ^{α} denotes the Riemann-Liouville like discrete fractional difference operator of order α is presented.

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1 Introduction

In this paper, we present the oscillatory behavior of forced nonlinear fractional difference equation of the form

$$\begin{aligned}\Delta^{\alpha}x(t)+f_1(t,x(t+\alpha)) =& v(t)+f_2(t,x(t+\alpha)), \quad t\in N_0, \quad 0<\alpha\leq 1,\\ \Delta^{\alpha-1}x(t)|_{t=0} =& x_0,\end{aligned} \tag{1}$$

where Δ^{α} is a Riemann-Liouville like discrete fractional difference, $f_i : [0,+\infty)\times R\to R, i=1,2$ and v are continuous with respect to t and $x, N_a=\{a,a+1,a+2,\ldots\}$.

Fractional differential equations have received considerable attention during recent years, because these equations are proved valuable tools for modeling of many phenomena in various fields. Fractional calculus finds its significant applications in the fields of viscoelasticity, capacitor theory, electrical circuits, electro-analytical chemistry, neurology, diffusion, control theory and statistics see [14], [15] and [16].

A rigorous theory of fractional differential equations has been started quite recently, see books [12], [13], and [15]. However, very little progress has been made to develop the theory of fractional difference equations see [7, 8, 9, 11]. In particular, nothing is known regarding the oscillatory behavior of (1) up to now. The study of Oscillation of fractional differential equations is initiated in [5, 17]. Motivated by [17] , we study the oscillation of fractional difference equations (1).

2 Definitions and Basic Lemmas

In this section, we introduce preliminary results of discrete fractional calculus.

Definition 2.1 *(see [2]) Let $\nu > 0$. The ν-th fractional sum f is defined by*

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s = a \mod (1)$ and $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \mod (1)$, and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on N_{a+v}.

Definition 2.2 *(see [2]) Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ-th fractional difference is defined as*

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t).$$

Theorem 2.3 *(see [3]) Let f be a real-value function defined on N_a and $\mu, \nu > 0$, then the following equalities hold:*

$$\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}[\Delta^{-\nu} f(t)];$$

$$\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a).$$

Lemma 2.4 *(see [2]) Let $\mu \neq 1$ and assume $\mu + \nu + 1$ is not a positive integer, then*

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

In order to discuss our results in Section 3, Now we state the following lemma [10] .

Lemma 2.5 *For $X \geq 0$ and $Y > 0$, we have*

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \quad \lambda > 1 \tag{2}$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0, \quad \lambda < 1, \tag{3}$$

where equality holds if and only if $X = Y$.

Lemma 2.6 *(see [2]) The equivalent fractional Taylor's difference formula of (1) is*

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))], t \in N_{\alpha}. \tag{4}$$

Proof: Apply the $\Delta^{-\alpha}$ operator to each side of (1), we obtain

$$\Delta^{-\alpha}\Delta^{\alpha}x(t) = \Delta^{-\alpha}[v(t) + f_2(t, x(t + \alpha)) - f_1(t, x(t + \alpha))] \tag{5}$$

Apply Theorem (2.3) to the left-hand side of (5),

$$\begin{aligned} \Delta^{-\alpha}\Delta^{\alpha}x(t) =& \Delta^{-\alpha}\Delta\Delta^{-(1-\alpha)}x(t) \\ =& \Delta\Delta^{-\alpha}\Delta^{-(1-\alpha)}x(t) - \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}x_0 \\ =& x(t) - \frac{x_0}{\Gamma(\alpha)}t^{(\alpha-1)}. \end{aligned}$$

Now, we apply Definition (2.1) to the right of (5) for $t \in N_a$, which yields (4) . This completes the proof.

3 Main Results

We consider the following conditions:

$$x f_i(t, x) > 0 \quad (i = 1, 2), \quad x \neq 0, \quad t \geq t_0 \tag{6}$$

and

$$|f_1(t, x)| \geq |p_1(t)| \, |x|^{\beta} \text{ and } |f_2(t, x)| \leq |p_2(t)| \, |x|^{\gamma}, \quad x \neq 0, \quad t \geq t_0, \tag{7}$$

where $p_1, p_2 \in C([t_0, \infty), R^+)$ and $\beta, \gamma > 0$ are real numbers.
Now we prove first theorem when $f_2 = 0$.

Theorem 3.1 *Suppose that condition (6) hold. If*

$$\lim_{t\to\infty}\inf t^{(1-\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}v(s)=-\infty, \tag{8}$$

and

$$\lim_{t\to\infty}\sup t^{(1-\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}v(s)=\infty, \tag{9}$$

then every solution of equation (1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (1) with $f_2 = 0$. Suppose that $T > t_0$ is large enough that $x(t) > 0$ for $t \geq T$.
Let $F(t) = v(t) + f_2(t, x(t+\alpha)) - f_1(t, x(t+\alpha))$, then we see from (4) that

$$x(t)\leq\frac{x_0}{\Gamma(\alpha)}t^{(\alpha-1)}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T-1}(t-s-1)^{(\alpha-1)}|F(s)|+\frac{1}{\Gamma(\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}v(s),\quad t\geq T,$$

$$\Gamma(\alpha)t^{(1-\alpha)}x(t)\leq x_0+t^{(1-\alpha)}\sum_{s=0}^{T-1}(t-s-1)^{(\alpha-1)}|F(s)|+t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}v(s),\quad t\geq T,$$

and hence

$$\Gamma(\alpha)t^{(1-\alpha)}x(t)\leq C(T)+t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}v(s),\quad t\geq T, \tag{10}$$

where

$$C(T)=x_0+\sum_{s=0}^{T-1}\left(\frac{T}{T-s-1}\right)^{(1-\alpha)}|F(s)|$$

and

$$\lim_{t\to\infty}C(t)=M<\infty,\quad t\geq T.$$

Taking the limit inferior of both sides of inequality (10) as $t \to \infty$, we get a contradiction to condition (8). This completes the proof of the theorem. Next we have the following results.

Theorem 3.2 *Suppose that conditions (6) and (7) hold with $\beta > 1$ and $\gamma = 1$. If*

$$\lim_{t\to\infty}\inf t^{(1-\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+H_\beta(s)\right]=-\infty, \tag{11}$$

and

$$\lim_{t\to\infty}\sup t^{(1-\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+H_\beta(s)\right]=\infty, \tag{12}$$

where

$$H_\beta(s) = (\beta-1)\beta^{\beta/(1-\beta)} p_1^{1/(1-\beta)}(s) p_2^{\beta/(\beta-1)}(s),$$

then every solution of equation (1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (4), say, $x(t) > 0$ for $t \geq T > t_0$. Using condition (7) in equation (4) with $\gamma = 1$ and $\beta > 1$ and $t \geq T$, we obtain

$$\begin{aligned}
\Gamma(\alpha)t^{(1-\alpha)}x(t) =& x_0 + t^{(1-\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))\right] \\
=& x_0 + t^{(1-\alpha)}\sum_{s=0}^{T-1}(t-s-1)^{(\alpha-1)}\left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))\right] \\
&+ t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))\right] \\
\leq& x_0 + t^{(1-\alpha)}\sum_{s=0}^{T-1}(t-s-1)^{(\alpha-1)}|F(s)| \\
&+ t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))\right] \\
\leq& C(T) + t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}\Big[v(s) \\
&+ f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha))\Big]
\end{aligned}$$

$$\begin{aligned}
\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq C(T) + t^{(1-\alpha)}\sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}\Big[v(s) \\
+ p_2(s)x(s+\alpha) - p_1(s)x^\beta(s+\alpha)\Big], \quad t \geq T.
\end{aligned} \tag{13}$$

We apply (2) in Lemma (2.5) with

$$\lambda = \beta, \quad X = p_1^{1/\beta}x \quad \text{and} \quad Y = \left(p_2 p_1^{-1/\beta}/\beta\right)^{1/(\beta-1)}$$

$$\begin{aligned}
\left(p_1^{1/\beta}(t)x(t+\alpha)\right)^\beta + (\beta-1)\left(p_2(t)p_1^{-1/\beta}(t)/\beta\right)^{\beta/(\beta-1)} \\
-\beta p_1^{1/\beta}(t)x(t+\alpha)\left(p_2(t)p_1^{-1/\beta}(t)/\beta\right) \geq 0
\end{aligned}$$

$$p_1(t)x^\beta(t+\alpha) + (\beta-1)\beta^{\beta/(1-\beta)}p_1^{1/(1-\beta)}(t)p_2^{\beta/(\beta-1)}(t) - p_2(t)x(t+\alpha) \geq 0$$

$$p_2(t)x(t+\alpha) - p_1(t)x^\beta(t+\alpha) \leq (\beta-1)\beta^{\beta/(1-\beta)}p_1^{1/(1-\beta)}(t)p_2^{\beta/(\beta-1)}(t), \quad t \geq T. \tag{14}$$

Using (14) in (13) , we obtain

$$\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}[v(s) + H_{\beta}(s)], \quad t \geq T. \tag{15}$$

Taking the limit inferior of both sides of inequality (15) as $t \to \infty$, we get a contradiction to condition (11). This completes the proof of the theorem.

Theorem 3.3 *Suppose that conditions (6) and (7) hold with* $\beta = 1$ *and* $\gamma < 1$*. If*

$$\liminf_{t\to\infty} t^{(1-\alpha)} \sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s) + H_{\gamma}(s)\right] = -\infty, \tag{16}$$

and

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s) + H_{\gamma}(s)\right] = \infty, \tag{17}$$

where

$$H_{\gamma}(s) = (1-\gamma)\gamma^{\gamma/(\gamma-1)}p_1^{\gamma/(\gamma-1)}(s)p_2^{1/(1-\gamma)}(s),$$

then every solution of equation (1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (4), say, $x(t) > 0$ for $t \geq T > t_0$. Using condition (7) in equation (4) with $\beta = 1$ and $\gamma < 1$, we get

$$\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}[v(s) + p_2(s)x^{\gamma}(s+\alpha) - p_1(s)x(s+\alpha)]. \tag{18}$$

Now we use (3) in Lemma (2.5) with

$$\lambda = \gamma, \quad X = p_2^{1/\gamma}x \quad \text{and} \quad Y = \left(p_1 p_2^{-1/\gamma}/\gamma\right)^{1/(\gamma-1)}$$

$$\left(p_2^{1/\gamma}(t)x(t+\alpha)\right)^{\gamma} - (1-\gamma)\left(p_1(t)p_2^{-1/\gamma}(t)/\gamma\right)^{\gamma/(\gamma-1)}$$
$$-\gamma p_2^{1/\gamma}(t)x(t+\alpha)(p_1(t)p_2^{-1/\gamma}(t)/\gamma) \leq 0$$

$$p_2(t)x^{\gamma}(t+\alpha) - p_1(t)x(t+\alpha) \leq (1-\gamma)\gamma^{\gamma/(1-\gamma)}p_1^{\gamma/(\gamma-1)}(t)p_2^{1/(1-\gamma)}(t) \quad t \geq T. \tag{19}$$

Using (19) in (18), we obtain

$$\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha}(t-s-1)^{(\alpha-1)}[v(s) + H_{\gamma}(s)], t \geq T. \tag{20}$$

Taking the limit inferior of both sides of inequality (20) as $t \to \infty$, we get a contradiction to condition (16). This completes the proof of the theorem. Finally we present the following more general result.

Theorem 3.4 *Suppose that conditions (6) and (7) hold with $\beta > 1$ and $\gamma < 1$. If*

$$\liminf_{t\to\infty} t^{(1-\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H_{\beta,\gamma}(s)\right] = -\infty, \tag{21}$$

and

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H_{\beta,\gamma}(s)\right] = \infty, \tag{22}$$

where

$$H_{\beta,\gamma}(s) = (\beta-1)\beta^{\beta/(1-\beta)}\xi^{\beta/(\beta-1)}(s)p_1^{1/(1-\beta)}(s)+(1-\gamma)\gamma^{\gamma/(1-\gamma)}\xi^{\gamma/(\gamma-1)}(s)p_2^{1/(1-\gamma)}(s)$$

with $\xi \in C([t_0,\infty), R^+)$, then every solution of equation (1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (1), say, $x(t) > 0$ for $t \geq T > t_0$. Using condition (7) in equation (4), we can write

$$\begin{aligned}\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq& C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s) \\ &+ t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[\xi(s)x(s+\alpha) - p_1(s)x^{\beta}(s+\alpha)\right] \\ &+ t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[p_2(s)x^{\gamma}(s+\alpha) - \xi(s)x(s+\alpha)\right], t \geq T.\end{aligned} \tag{23}$$

We may bound the terms $(\xi x - p_1 x^{\beta})$ and $(p_2 x^{\gamma} - \xi x)$ by using the inequalities (14) with $p_2 = \xi$, and (19) with $p_1 = \xi$ respectively, we get

$$\begin{aligned}\Gamma(\alpha)t^{(1-\alpha)}x(t) \leq& C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} v(s) \\ &+ t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Big[\Big[(\beta-1)\beta^{\beta/(1-\beta)}\xi^{\beta/(\beta-1)}(s)p_1^{1/(1-\beta)}(s)\Big] \\ &+ \Big[(1-\gamma)\gamma^{\gamma/(1-\gamma)}\xi^{\gamma/(\gamma-1)}(s)p_2^{1/(1-\gamma)}(s)\Big]\Big]\end{aligned}$$

$$\Gamma(\alpha)t^{1-\alpha}x(t) \leq C(T) + t^{(1-\alpha)} \sum_{s=T}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H_{\beta,\gamma}(s)\right], t \geq T. \tag{24}$$

Taking the limit inferior of both sides of inequality (24) as $t \to \infty$, we get a contradiction to condition (21). This completes the proof of the theorem.
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