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Optimal Inequalities for Seiffert and $K_r(a, b)$ Means

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Abstract

Given two positive real numbers a and b, let A(a, b), G(a, b) and P(a, b) denote their arithmetic mean, geometric mean and Seiffert mean respectively. Let $K_r(a,b) = \sqrt[r]{\frac{2}{3}A^r(a,b) + \frac{1}{3}G^r(a,b)}$ for $r > 0$. In this paper, we find the greatest value α and the least value β such that the double inequality

$$K_\alpha(a,b) < P(a,b) < K_\beta(a,b)$$

holds for all $a,\ b > 0$ with $a \neq b$.

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1 Introduction

In this paper we consider several means of two positive real numbers a and b. Recall that the arithmetic mean and the geometric mean are defined by $A(a,b) = \frac{a+b}{2}$ and $G(a,b) = \sqrt{ab}$. The family$(K_r(a,b))_{r>0}$ of means of a and b is defined by[7]

$$K_r(a,b) = \sqrt[r]{\frac{2A^r(a,b) + G^r(a,b)}{3}}.$$

For $a, b > 0$ with $a \neq b$ the Seiffert mean $P(a,b)$ was introduced by Seiffert [8] as follows:

$$P(a,b) = \frac{a-b}{4\arctan(\sqrt{a/b}) - \pi}.$$

Recently, both means have been the subject of intensive research [1-14] and therein.

The following bounds for the Seiffert mean $P(a,b)$ in terms of the power mean $M_r(a,b) = ((a^r + b^r)/2)^{1/r} (r \neq 0)$ were presented by Jagers in [13]:

$$M_{1/2} < P(a,b) < M_{2/3}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Hästö [14] found the sharp lower bound for the Seiffert mean as follow:

$$M_{log2/log\pi}(a,b) < P(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [9], Seiffert proved

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b) + 2G(a,b)} \quad \text{and} \quad P(a,b) > \frac{2}{\pi}A(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [10] it proved that

$$I(a,b) < K_2(a,b)$$

for all positive real numbers $a \neq b$, where $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ is the identric mean of two positive real numbers a and b with $a \neq b$.

J.Sàndor proved in [5] that

$$I(a,b) > K_1(a,b)$$

for all positive real numbers $a \neq b$.

In [7], the author proved that

$$K_{\frac{6}{5}}(a,b) < I(a,b) < K_{\frac{\ln 3 - \ln 2}{1 - \ln 2}}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to find the greatest value α and the least value β such that the inequality

$$K_\alpha(a,b) < P(a,b) < K_\beta(a,b)$$

holds for all $a,\ b > 0$ with $a \neq b$.

2 Main Results

Lemma 2.1 *For $x > 1$ and $r \in (0,1)$, let*

$$\begin{aligned} h(x) &= x(x + 2x^{r-1})^2 - (x + 2x^{r+1} + 2rx^{r+1} - 2rx^{r-1})(x^2 + 2x^r) \\ &+ (x^2 - 1)(1 + 2x^r)(2x + 2rx^{r-1}). \end{aligned} \tag{2.1}$$

We have:

case 1. If $r = \frac{4}{5}$ then $h(x) > 0$ holds for $x \in [1, +\infty)$.

case 2. If $r = \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2}$ then there exists $\lambda \in (1, +\infty)$ such that $h(x) < 0$ for $x \in [1, \lambda)$ and $h(x) > 0$ for $x \in (\lambda, +\infty)$.

proof.

$$\begin{aligned} h(x) &= x(x+2x^{r-1})^2-(x+2x^{r+1}+2rx^{r+1}-2rx^{r-1})(x^2+2x^r) \\ &+ (x^2-1)(1+2x^r)(2x+2rx^{r-1}) \\ &= 2xh_1(x), \end{aligned} \tag{2.2}$$

where

$$h_1(x)=x^2-1+(1-r)x^{r+2}-2x^{2r}+(2r-1)x^r+2x^{2r-2}-rx^{r-2}. \tag{2.3}$$

Then simple computations lead to

$$\lim_{x\to 1^+} h_1(x)=0, \tag{2.4}$$

$$\lim_{x\to +\infty} h_1(x)=+\infty. \tag{2.5}$$

$$h_1'(x)=xh_2(x), \tag{2.6}$$

$$\begin{aligned} h_2(x)= \quad & 2+(1-r)(2+r)x^r-4rx^{2r-2}+(2r-1)rx^{r-2} \\ & +2(2r-2)x^{2r-4}-r(r-2)x^{r-4}, \end{aligned} \tag{2.7}$$

$$\lim_{x\to 1^+} h_2(x)=0, \tag{2.8}$$

$$\lim_{x\to +\infty} h_2(x)=+\infty. \tag{2.9}$$

$$h_2'(x)=x^{r-5}h_3(x), \tag{2.10}$$

$$\begin{aligned} h_3(x)= \quad & (2r-r^2-r^3)x^4-(8r^2-8r)x^{r+2}+(2r^3-5r^2+2r)x^2 \\ & +(8r^2-24r+16)x^r-(r^3-6r^2+8r), \end{aligned} \tag{2.11}$$

$$\lim_{x\to 1^+} h_3(x)=16-20r, \tag{2.12}$$

$$\lim_{x\to +\infty} h_3(x)=+\infty. \tag{2.13}$$

$$h_3'(x)=2xh_4(x), \tag{2.14}$$

$$\begin{aligned} h_4(x)= \quad & 2(2r-r^3-r^2)x^2-(4r^2-4r)(2+r)x^r+(2r^3-5r^2+2r) \\ & +(4r^3-12r^2+8r)x^{r-2}, \end{aligned} \tag{2.15}$$

$$\lim_{x\to 1^+} h_4(x)=r(22-23r). \tag{2.16}$$

$$h_4'(x)=4rxh_5(x), \tag{2.17}$$

$$h_5(x)=2-r^2-r-(r^2-r)(r+2)x^{r-2}+(r^2-3r+2)(r-2)x^{r-4}, \tag{2.18}$$

$$\lim_{x\to 1^+} h_5(x)=(7r-2)(1-r), \tag{2.19}$$

$$\lim_{x\to +\infty} h_5(x)=2-r-r^2. \tag{2.20}$$

$$h_5'(x)=(1-r)(2-r)x^{r-5}h_6(x), \tag{2.21}$$

$$h_6(x) = -r(r+2)x^2 + (r-2)(r-4) \tag{2.22}$$

$$\lim_{x\to 1^+} h_6(x) = 8 - 8r, \tag{2.23}$$

$$\lim_{x\to +\infty} h_6(x) = -\infty, \tag{2.24}$$

$$h_6'(x) = -2r(r+2)x < 0,$$

and $h_6(x)$ is strictly decreasing in $[1, +\infty)$. Combining (2.23) and (2.24) we can get that there exist $x_1 > 1$ such that $h_6(x) > 0$ for $x \in [1, x_1]$ and $h_6(x) < 0$ for $x \in [x_1, +\infty)$. This implies that $h_5(x)$ is strictly increasing in $[1, x_1]$ and strictly decreasing in $[x_1, +\infty)$.

From (2.19), (2.20) and the monotonicity of $h_5(x)$ we clearly see that $h_5(x) > 0$ for $x > 1$ and $r \in \{\frac{4}{5}, \frac{\ln 3-\ln 2}{\ln \pi-\ln 2}\}$. Hence $h_4(x)$ is strictly increasing in $[1, +\infty)$, together with (2.16) and (2.14) we can get $h_4(x) > 0$ and $h_3'(x) > 0$ for $x > 1$. Obviously $h_3(x)$ is strictly increasing in $[1, +\infty)$.

Now we divide the proof into two cases.

Case 1. If $r = \frac{4}{5}$.

From (2.12) and the monotonicity of $h_3(x)$ we have $h_3(x) > 0$ for $x \in [1, +\infty)$, hence we know that $h_2(x)$ is strictly increasing in $[1, +\infty)$.

From (2.8) and the monotonicity of $h_2(x)$ we can get $h_2(x) > 0$ for $x \in [1, +\infty)$, so we know that $h_1(x)$ is strictly increasing in $[1, +\infty)$.

From (2.4) and the monotonicity of $h_1(x)$ we have $h_1(x) > 0$ for $x \in [1, +\infty)$, hence we know that $h(x)$ is strictly increasing in $[1, +\infty)$.

Case 2. If $r = \frac{\ln 3-\ln 2}{\ln \pi-\ln 2}$.

From (2.12) and (2.13) together with the monotonicity of $h_3(x)$ we clearly see that there exists $\lambda_1 \in [1, +\infty)$ such that $h_3(x) < 0$ for $x \in [1, \lambda_1)$ and $h_3(x) > 0$ for $x \in (\lambda_1, +\infty)$, hence we know that $h_2(x)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

It follows from (2.8) and (2.9) together with the monotonicity of $h_2(x)$ in $[1, \lambda_1]$ and in $[\lambda_1, +\infty)$ that there exists $\lambda_2 > \lambda_1$ such that $h_2(x) < 0$ for $x \in [1, \lambda_2)$ and $h_2(x) > 0$ for $x \in [\lambda_2, +\infty)$, hence we know that $h_1(x)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$.

From (2.4) and (2.5) together with the monotonicity of $h_1(x)$ in $[1, \lambda_2]$ and in $[\lambda_2, +\infty)$, we clearly see that there exists $\lambda > \lambda_2$ such that $h_1(x) < 0$ for $x \in [1, \lambda]$ and $h_1(x) > 0$ for $x \in [\lambda, +\infty)$, hence from (2.2) we know that $h(x) < 0$ for $x \in [1, \lambda]$ and $h(x) > 0$ for $x \in [\lambda, +\infty)$.

Theorem 2.2 *The double inequality*

$$K_\alpha(a, b) < P(a, b) < K_\beta(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{4}{5}$ and $\beta \geq \frac{\ln 3-\ln 2}{\ln \pi-\ln 2}$.

Proof. Firstly, we prove

$$K_{\frac{4}{5}}(a,b) < P(a,b), \tag{2.25}$$

$$P(a,b) < K_{\frac{\ln 3-\ln 2}{\ln \pi-\ln 2}}(a,b) \tag{2.26}$$

for all $a,b>0$ with $a \neq b$.

Let $a=e^t, b=e^{-t}, t>0$, Then

$$K_r(a,b)/P(a,b) = \{\frac{2(\frac{e^t+e^{-t}}{2})^r+1}{3}\}^{\frac{1}{r}}/\{\frac{e^t-e^{-t}}{4\arctan e^t-\pi}\}. \tag{2.27}$$

Let

$$f(t) = \ln(K_r(a,b)/P(a,b)) = \frac{1}{r}\ln\frac{2ch^r t+1}{3} - \ln(e^t-e^{-t}) + \ln(4\arctan e^t-\pi). \tag{2.28}$$

Simple computations lead to

$$\lim_{t\to 0^+} f(t) = 0, \tag{2.29}$$

$$\lim_{t\to +\infty} f(t) = \ln[\frac{\pi}{2}(\frac{2}{3})^{\frac{1}{r}}], \tag{2.30}$$

$$f'(t) = \frac{-(2ch^{r-1}t+cht)(1+e^{2t})}{(1+2ch^r t)sht(4\arctan e^t-\pi)(1+e^{2t})}f_1(t), \tag{2.31}$$

where

$$f_1(t) = 4\arctan e^t - \pi - 2\frac{sht+2shtch^r t}{ch^2 t+2ch^r t}. \tag{2.32}$$

$$\lim_{t\to 0^+} f_1(t) = 0, \tag{2.33}$$

$$\lim_{t\to +\infty} f_1(t) = \pi, \tag{2.34}$$

$$f_1'(t) = \frac{2}{(ch^2 t+2ch^r t)^2}h(cht), \tag{2.35}$$

where

$$\begin{aligned} h(x) &= x(x+2x^{r-1})^2 - (x+2x^{r+1}+2rx^{r+1}-2rx^{r-1})(x^2+2x^r) \\ &+ (x^2-1)(1+2x^r)(2x+2rx^{r-1}),\ x>1. \end{aligned}$$

If $r=\frac{4}{5}$, from (2.35) and the Lemma we know that $f_1'(t)>0$ for $t\in(0,+\infty)$, hence $f_1(t)$ is strictly increasing in $(0,+\infty)$. From (2.31), (2.32) and the monotonicity of $f_1(t)$ we get $f'(t)<0$ for $t\in(0,+\infty)$ and $f(t)$ is strictly decreasing in $t\in(0,+\infty)$. Using (2.29) and the monotonicity of $f(t)$ we have $f(t)<0$ in $t\in(0,+\infty)$. So (2.25) is proved.

If $r=\frac{\ln 3-\ln 2}{\ln \pi-\ln 2}$, from (2.35) and the Lemma we know that there exists $t_1\in(0,+\infty)$ such that $f_1'(t)<0$ for $t\in(0,t_1)$ and $f_1'(t)>0$ for $t\in(t_1,+\infty)$, hence $f_1(t)$ is strictly decreasing in $(0,t_1)$ and strictly increasing in $(t_1,+\infty)$. From (2.33) and (2.34) together with the monotonicity of $f_1(t)$ in $(0,t_1)$ and in $(t_1,+\infty)$, we know that there exists $t_2>t_1$ such that $f_1(t)<0$ for $t\in(0,t_2)$ and $f_1(t)>0$ for $t\in(t_2,+\infty)$, hence with (2.31) we clearly see that $f'(t)>0$ for $t\in(0,t_2)$ and $f'(t)<0$ for $t\in(t_2,+\infty)$, and from (2.29) and (2.30) we know that $f(t)>0$ for $t\in(0,+\infty)$. So $K_{\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}}(a,b)>p(a,b)$ holds for all $a,b>0$ with $a\neq b$, (2.26) is proved.

Secondly, we prove that the parameters $\frac{4}{5}$ and $\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}$ cannot be improved in each inequality.

For any $0<\varepsilon<\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}$ and $x>1$, we have

$$\lim_{x\to+\infty}\frac{K_{\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}-\epsilon}(x,1)}{P(x,1)}=\frac{\pi}{2}\left(\frac{2}{3}\right)^{\frac{1}{\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}-\varepsilon}}<1. \tag{2.36}$$

Inequality (2.36) implies that for any $0<\varepsilon<\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}$ there exists $X=X(\varepsilon)>1$ such that $K_{\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}-\epsilon}(x,1)<P(x,1)$ for $x\in(X,+\infty)$. Hence the parameter $\frac{\ln 3-\ln 2}{\ln\pi-\ln 2}$ in the right-side inequality cannot be improved, it is the best possible.

Next we prove the parameter $\frac{4}{5}$ in the left-side inequality cannot be improved.

For any $x>1$, if $r>\frac{4}{5}$, then from (2.12) and the continuity of $h_3(x)$ we see that there exists $\delta=\delta(r)>0$ such that

$$h_3(x)<0,\quad \forall x\in[1,1+\delta).$$

Then (2.8) and (2.10) imply that

$$h_2(x)<0,\quad \forall x\in[1,1+\delta).$$

From (2.2), (2.4) and (2.6) we have

$$h(x)<0,\quad \forall x\in[1,1+\delta).$$

Then from (2.33) and (2.35) we see that there exists $\delta'=\delta'(\delta)>0$ such that

$$f_1(t)<0,\quad \forall t\in[0,\delta').$$

Then (2.29) and (2.31) imply that

$$f(t)>0,\quad \forall t\in[0,\delta').$$

Therefore, $P(e^t,e^{-t})<K_{\frac{4}{5}}(e^t,e^{-t})$ for $t\in[0,\delta')$. Hence the parameter $\frac{4}{5}$ in the left-side inequality cannot be improved, it is the best possible.

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