On Weakly Concircular Symmetries of a Generalized Sasakian Space Form

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Abstract

In the present paper, we have studied weakly concircular symmetric, weakly Concircular Ricci symmetric and special weakly Ricci symmetric of generalized Sasakian space form.

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1 Introduction

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian space form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmostu and Cosymplectic space form. In order to generalize such space form in a common frame, P. Alegre, D. E. Blair and A. Carriazo [1], introduced the notion of generalized Sasakian space form. It is defined as an almost Contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor is given

by,

$$
\begin{aligned}
R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\
&+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},
\end{aligned}
$$

for any vector fields X,Y,Z on M, where f_1, f_2, f_3 are differentiable function on M. In [1], authors have given several examples of such manifolds. If $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$, then generalized Sasakian space form with Sasakian structure becomes Sasakian space form.

As a generalization of Chaki's pseudosymmetric and pseudo Ricci symmetric manifolds [5, 6], the notion of weakly symmetric manifolds were introduced by Tamassy and Binh [16, 17]. A non-flat Riemannian manifold M is called a weakly symmetric manifold if its curvature tensor R of type $(0,4)$ satisfies the condition

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \quad (1) \\
&+ H(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\
&+ E(V)R(Y,Z,U,X),
\end{aligned}
$$

for all vector fields $X,Y,Z,U,V \in \chi(M^{2n+1})$, where A, B, H, D and E are associated 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g.

In 1999 De and Bandyopadhyay [9] studied a weakly symmetric manifolds and proved that in such a manifold the associated 1-forms $B = H$ and $D = E$. Hence (1) reduces to the following:

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \quad (2) \\
&+ B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\
&+ D(V)R(Y,Z,U,X).
\end{aligned}
$$

Shaikh A. A. and Hui S. K. [12] introduced the notion of weakly Concircular symmetric manifolds. Again Tamassy and Binh [16, 17] introduced the notion of weakly Ricci symmetric manifolds. In [7] De and Ghosh introduced the notion of weakly Concircular Ricci symmetric manifolds. In [13] and [10] authors have studied weak concircular symmetries of Kenmotsu manifold and Lorentzian Concircular Structure Manifolds. In [15] the authors have studied weakly symmetric and weakly Ricci-symmetric Lorentzian β-Kenmotsu manifold.

In [11], C. Özgür studied weakly symmetric Kenmotsu manifolds. The notion of special weakly Ricci symmetric manifolds was introduced and studied

by H. Singh, and Q. Khan in [14]. Special weakly Ricci symmetric Kenmotsu manifolds is studied by Nesip Aktan et al. [3]. Motivated by these ideas, in the present paper we made an attempt to study weakly Concircular symmetric and special weakly Ricci symmetric generalized Sasakian space form. The present paper is organized as follows:

In section 2, we recall some preliminary results. In section 3, we studied weakly Concircular symmetric generalized Sasakian space form and obtain all 1-forms A, B and D of a weakly concircular symmetric generalized Sasakian space form provided $(1 - 2n)f_3 - 3f_2 \neq 0$. And in section 4, we have studied weakly Concircular Ricci symmetric generalized Sasakian space form and obtain all 1-forms A, B and D of a weakly concircular Ricci symmetric generalized Sasakian space form provided $(1 - 2n)f_3 - 3f_2 \neq 0$. Also it is proved that the sum of the associated 1-forms of a weakly Concircular Ricci symmetric generalized Sasakian space form is non-vanishing, provided that $(1-2n)f_3-3f_2 \neq 0$. Finally section 5, we have studied special weakly Ricci symmetric generalized Sasakian space form.

2   Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost Contact manifold if there exists on M, a $(1, 1)$ tensor field ϕ, a vector field ξ and a 1-form η [4] such that,

$$\phi^2(X) = -X + \eta(X)\xi, \tag{3}$$

$$\eta(\phi X) = 0, \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5}$$

$$\phi\xi = 0, \quad \eta(\xi) = 0, \quad g(X, \xi) = \eta(X), \tag{6}$$

for any vector fields X, Y on M.

If in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well known that a Contact metric manifold is a K-contact manifold if and only if $(\nabla_X \xi) = -\phi(X)$ for any vector field X on M.

Given an almost Contact metric manifold (M, ϕ, ξ, η, g) we say that M is an generalized Sasakian space form [1], if there exists three functions f_1, f_2 and f_3 on M such that

$$
\begin{aligned}
R(X, Y)Z \;=\; & f_1\{g(Y, Z)X - g(X, Z)Y\} \\
+\; & f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
+\; & f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+\; & g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},
\end{aligned}
\tag{7}
$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian space form $M(C)$, which can be obtained as particular cases of generalized Sasakian space form by taking $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$. Further in a $(2n + 1)$-dimensional generalized Sasakian space form, we have [1], [2]

$$(\nabla_X \phi)(Y) = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X), \tag{8}$$

$$(\nabla_X \xi) = -(f_1 - f_3)\phi(X), \tag{9}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{10}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) \tag{11}$$
$$- (3f_2 + (2n-1)f_3)\eta(X)\eta(Y),$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{12}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{14}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{15}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{16}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{17}$$

Definition 2.1 *A Riemannian manifold is called weakly Concircular symmetric manifold if its Concircular curvature tensor C of type $(0, 4)$ is not identically zero and satisfies the condition [12]*

$$(\nabla_X C)(Y,Z,U,V) = A(X)C(Y,Z,U,V) + B(Y)C(X,Z,U,V) \tag{18}$$
$$+ B(Z)C(Y,X,U,V) + D(U)C(Y,Z,X,V)$$
$$+ D(V)C(Y,Z,U,X),$$

where A, B and D are 1-forms (not simultaneously zero) and C is given by,

$$C(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{2n(2n+1)}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)], \tag{19}$$

where r is the scalar curvature of the manifold.

Definition 2.2 *A Riemannian manifold is called weakly Concircular Ricci symmetric manifold [7] if its Concircular Ricci tensor K of type $(0, 2)$ is not identically zero and satisfies the condition*

$$(\nabla_X K)(Y,Z) = A(X)K(Y,Z) + B(Y)K(X,Z) + D(Z)K(Y,X), \tag{20}$$

where A, B and D are 1-forms (not simultaneously zero) and Concircular Ricci tensor K is given by

$$K(Y,V) = S(Y,V) - \frac{r}{2n+1}g(Y,V). \tag{21}$$

Definition 2.3 *A Riemannian manifold is called a special weakly Ricci-symmetric manifold [14] if*

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \qquad (22)$$

for any vector fields X and Y on M, where A is a 1-form and is defined by

$$A(X) = g(X, \rho), \qquad (23)$$

where ρ is the associated vector field.

3 Weakly Concircular Symmetric Generalized Sasakian space form

Setting $Y = V = e_i$ in (18) and taking summation over $i, 1 \leq i \leq n$, we get

$$
\begin{aligned}
&(\nabla_X S)(Z, U) - \frac{dr(X)}{2n+1} g(Z, U) \qquad (24) \\
=\;& A(X)[S(Z, U) - \frac{r}{2n+1} g(Z, U)] + B(Z)[S(X, U) - \frac{r}{2n+1} g(X, U)] \\
+\;& D(U)[S(X, Z) - \frac{r}{2n+1} g(X, Z)] + B(R(X, Z)U) + D(R(X, U)Z) \\
-\;& \frac{r}{2n(2n+1)}[B(X) + D(X)g(Z, U) - B(Z)g(X, U) - D(U)g(Z, X)].
\end{aligned}
$$

Substituting $X = Z = U = \xi$ in (24) and then using (13) and (16) we get

$$A(\xi) + B(\xi) + D(\xi) = \frac{(1-2n)df_3 - 3df_2}{(1-2n)f_3 - 3f_2}, \qquad (25)$$

provided $(1 - 2n)f_3 - 3f_2 \neq 0$.
Hence we state,

Theorem 3.1 *In a weakly Concircular symmetric generalized Sasakian space form the relation (25) holds, provided $(1 - 2n)f_3 - 3f_2 \neq 0$.*

Suppose $(1 - 2n)f_3 - 3f_2$ is a non zero constant, then (25) reduces to

$$A(\xi) + B(\xi) + D(\xi) = 0.$$

Thus we have,

Lemma 3.2 *In a weakly Concircular symmetric generalized Sasakian space form the sum of 1-forms is zero everywhere if and only if $(1 - 2n)f_3 - 3f_2$ is a non zero constant.*

Now substituting X and Z by ξ in (24) and the using (13), (14) and (16), we get

$$[A(\xi) + B(\xi)][2n(2n+1)(f_1 - f_3) - r]\eta(U) \tag{26}$$
$$+ \quad \frac{2n(2n+1)(f_1 - f_3) - r}{2n}[(2n-1)D(U) + \eta(U)D(\xi)]$$
$$- \quad [2n(2n+1)(df_1 - df_3) - dr] = 0,$$

By virtue of (25), it follows from (26) that

$$D(U) = [D(\xi) - \frac{(1-2n)df_3 - 3df_2}{(1-2n)f_3 - 3f_2}]\eta(U), \tag{27}$$

provided $(1-2n)f_3 - 3f_2 \neq 0$.

Again setting $X = U = \xi$ in (24) and proceeding in a similar manner as above, we get

$$B(Z) = [B(\xi) - \frac{(1-2n)df_3 - 3df_2}{(1-2n)f_3 - 3f_2}]\eta(Z), \tag{28}$$

provided $(1-2n)f_3 - 3f_2 \neq 0$.

Similarly setting $Z = U = \xi$ in (24) and using (13) and (16) we obtain

$$A(X) \quad = \quad \frac{(1-2n)df_3 - 3df_2}{(1-2n)f_3 - 3f_2} - \frac{1}{2n}[B(X) + D(X)] \tag{29}$$
$$- \quad \frac{2n-1}{(1-2n)f_3 - 3f_2}[B(\xi) + D(\xi)]\eta(Z),$$

Thus,

Theorem 3.3 *In a weakly Concircular symmetric generalized Sasakian space form, the associated* 1-*form* D, B *and* A *are given by the relation (27), (28) and (29), provided* $(1-2n)f_3 - 3f_2 \neq 0$.

4   Weakly Concircular Ricci-symmetric generalized Sasakian space form

In view of (21), (20) can be written as

$$(\nabla_X S)(Y, Z) - \frac{dr(X)}{2n+1}g(Y, Z) \quad = \quad A(X)[S(Y, Z) - \frac{r}{2n+1}g(Y, Z)] \tag{30}$$
$$+ \quad B(Y)[S(X, Z) - \frac{r}{2n+1}g(X, Z)]$$
$$+ \quad D(Z)[S(Y, X) - \frac{r}{2n+1}g(Y, X)].$$

Setting $X = Y = Z = \xi$ in (30), we get the relation (25) and hence we can state the following,

Theorem 4.1 *In a weakly Concircular Ricci-symmetric generalized Sasakian space form, the relation (25) holds, provided $(1 - 2n)f_3 - 3f_2 \neq 0$.*

Now, Substituting X and Y by ξ in (30), we get

$$D(Z) = D(\xi)\eta(Z), \qquad (1 - 2n)f_3 - 3f_2 \neq 0. \qquad (31)$$

Next, Substituting X and Z by ξ in (30), we get

$$B(Y) = B(\xi)\eta(Y), \qquad (1 - 2n)f_3 - 3f_2 \neq 0. \qquad (32)$$

Again, Substituting Y and Z by ξ in (30) and using (16) and (25), we get

$$A(X) = \frac{(1 - 2n)df_3 - 3df_2}{(1 - 2n)f_3 - 3f_2} + [A(\xi) - \frac{(1 - 2n)df_3 - 3df_2}{(1 - 2n)f_3 - 3f_2}]\eta(X), \qquad (33)$$

provided $(1 - 2n)f_3 - 3f_2 \neq 0$.
Hence we can state,

Theorem 4.2 *In a weakly Concircular Ricci-symmetric generalized Sasakian space form, the associated 1-forms D, B and A are given by (31), (32) and (33) respectively provided $(1 - 2n)f_3 - 3f_2 \neq 0$.*

Adding (31), (32) and (33) and using (25), we get

$$A(X) + B(X) + D(X) = \frac{(1 - 2n)df_3 - 3df_2}{(1 - 2n)f_3 - 3f_2}, \qquad (1 - 2n)f_3 - 3f_2 \neq 0. \quad (34)$$

Hence we state,

Theorem 4.3 *In a weakly Concircular Ricci-symmetric generalized Sasakian space form, the sum of the associated 1-forms is given by (34).*

Also from (34), we state the following,

Lemma 4.4 *There exist no weakly Concircular Ricci symmetric generalized Sasakian space form of constant scalar curvature, unless the sum of the associated 1-forms is everywhere zero.*

5   Special Weakly Ricci-Symmetric Generalized Sasakian space form

Let us assume that (22) and (23) is satisfied by generalized Sasakian space form. Taking the cyclic sum in (22), we get

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \qquad (35)$$
$$= 4[A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X)],$$

If M admits cyclic Ricci tensor then (35) reduces to

$$A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) = 0. \tag{36}$$

Taking $Z = \xi$ in (36) and using (16) and (23), we get

$$2n(f_1 - f_3)A(X)\eta(Y) + 2n(f_1 - f_3)A(Y)\eta(X) + \eta(\rho)S(X,Y) = 0. \tag{37}$$

Now putting $Y = \xi$ in (37) and using (3), (16) and (23), we get

$$2n(f_1 - f_3)[A(X) + 2\eta(\rho)\eta(X)] = 0. \tag{38}$$

Again putting $X = \xi$ in (38), we get

$$2n(f_1 - f_3)\eta(\rho) = 0. \tag{39}$$

Using (39) in (38), we get $(f_1 - f_3)A(X) = 0$, for any vector field X on M. Thus we state:

Theorem 5.1 *If a special weakly Ricci-symmetric generalized Sasakian space form admits a cyclic parallel Ricci tensor then the* 1-*form A must vanish provided $f_1 - f_3 \neq 0$.*

For an Einstein manifold $(\nabla_X S)(Y,Z) = 0$ and $S(Y,Z) = kg(Y,Z)$, then (22) gives

$$2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) = 0. \tag{40}$$

Taking $Z = \xi$ in (40) and using (16) and (23), we get

$$4n(f_1 - f_3)A(X)\eta(Y) + 2n(f_1 - f_3)A(Y)\eta(X) + \eta(\rho)S(X,Y) = 0. \tag{41}$$

Taking $X = \xi$ in (41) and using (3), (16) and (23), we get

$$4n(f_1 - f_3)\eta(\rho)\eta(Y) + 2n(f_1 - f_3)[A(Y) + \eta(\rho)\eta(Y)] = 0. \tag{42}$$

Taking $Y = \xi$ in (42) and using (16) and (23), we get

$$4n(f_1 - f_3)\eta(\rho) = 0. \tag{43}$$

Using (43) in (42), we get $(f_1 - f_3)A(Y) = 0$, for any vector field Y on M. Thus we state,

Theorem 5.2 *A special weakly Ricci-symmetric generalized Sasakian space form can not be an Einstein manifold if the* 1-*form $A \neq 0$, provided $f_1 - f_3 \neq 0$.*

Taking $Z = \xi$ in (22), we have

$$(\nabla_X S)(Y, \xi) = 2A(X)S(Y, \xi) + A(Y)S(X, \xi) + \eta(\rho)S(Y, X). \qquad (44)$$

This implies,

$$
\begin{aligned}
&\nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi) \qquad (45)\\
&= 2A(X)2n(f_1 - f_3)\eta(Y) + A(Y)2n(f_1 - f_3)\eta(X) + \eta(\rho)S(Y, X).
\end{aligned}
$$

Taking $Y = \xi$ in (45) and using (3), (9), (16) and (23), we get

$$4n(f_1 - f_3)[A(X) + \eta(\rho)\eta(X)] = 0. \qquad (46)$$

Putting $X = \xi$ in (46), we obtain

$$(f_1 - f_3)\eta(\rho) = 0. \qquad (47)$$

Using (47) in (46) we get

$$2n(f_1 - f_3)A(X) = 0, \qquad (48)$$

for any vector fields Y on M. Hence in view of (48), from (22) we get $\nabla_X S = 0$, Hence we state:

Theorem 5.3 *The Ricci tensor of a special weakly Ricci-symmetric generalized Sasakian space form is parallel, provided $f_1 - f_3$ is a non zero constant.*

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