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On W. Gordon's integral (1929) and related identities

Nasser Saad

Department of Mathematics and Statistics, University of Prince Edward Island
Charlottetown, Prince Edward Island C1A 4P3, Canada.

Abstract

Analytic evaluation of Gordon's integral

$$\mathrm{J}_c^{j(\pm p)}(b,b';\lambda,w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(b';c\pm p;zx)dx,$$

are given along with convergence conditions. It shows enormous number of definite integrals, frequently appear in theoretical and mathematical physics applications, easily deduced from this generalized integral.

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1 Introduction

Among the important integrals in theoretical and mathematical physics is W. Gordon's integral [3], see also [4, 5, 8, 10],

$$\mathrm{J}_c^{j(\pm p)}(b,b';\lambda,w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(b';c\pm p;zx)dx$$
$$(c+j>0;\ \lambda>0;\ c,c\pm p\neq 0,-1,-2,\dots;\ p\geq 0;\ j=0,\pm1,\pm2,\dots),\quad (1)$$

where ${}_1F_1$ is the confluent hypergeometric function

$${}_1F_1(b;c;z)=\sum_{k=0}^\infty\frac{(b)_k}{[(c)_k}\frac{z^k}{k!}$$

in which $(b)_k=b(b+1)\dots(b+n-1)=\Gamma(b+k)/\Gamma(b)$ is the Pochhammer symbol defined in terms of Gamma function. The massive uses of this integral and the subclasses of it span large volume of research papers and monographs [4, 5, 6, 8, 10]. It was proven (Lemma 1 in [8]) that, for $c+j>0$ and $|w|+|z|<|\lambda|$,

$$\mathrm{J}_c^{j(\pm p)}(b,b';\lambda,w,z)=\frac{\Gamma(c+j)}{\lambda^{c+j}}\,F_2\left({c+j;\ \ b,\ \ \ b' \atop \qquad c,\ \ c\pm p};\frac{w}{\lambda},\frac{z}{\lambda}\right),\tag{2}$$

where the second Appell function reads ([1], equation (2))

$$F_2\left({a;b,b' \atop c,c'};w,z\right)\equiv F_2\left(a;b,b';c,c';w,z\right)=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+p}\,(b)_m\,(b')_p}{(c)_m\,(c')_p}\frac{w^m\,z^p}{m!\,p!},$$
$$(c,c'\neq 0,-1,\ldots;|w|+|z|<1).\tag{3}$$

Exact analytical expressions of this integral by means of more elementary functions are given in the present work, where many subclasses are analysed and evaluated in simplified expressions allow for faster computations.

2 Closed form expressions

By means of the double integral representation of the second Appell function ([1], equation 7; see also [8]), for $c, c-p\neq 0,-1,-2,\ldots$, $j,p=0,1,2,\ldots$, $|w|+|z|<1$, it follows, for $j\geq p$, that

$$\begin{aligned}\mathrm{J}_c^{j(\pm p)}(b,b';\lambda,w,z)&=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(b';c\pm p;zx)dx\\&=\frac{\Gamma(c+j)}{\lambda^{c+j-b'}(\lambda-z)^{b'}}\sum_{k=0}^{j\mp p}\frac{(-j\pm p)_k(b')_k}{(c\pm p)_k\ k!}\left(1-\frac{\lambda}{z}\right)^{-k}\\&\times F_1\left(b,c+j-b',b'+k;c;\frac{w}{\lambda},\frac{w}{\lambda-z}\right),\end{aligned}$$
$$(c+j>0;\ \lambda>0;\ c,c\pm p\neq 0,-1,\ldots;\ p\geq 0;\ j=0,\pm1,\ldots;\ |w|+|z|<\lambda),\tag{4}$$

particularly, for $p = j = 0, 1, 2, \ldots,$

$$\begin{aligned}
\mathrm{J}_c^{jj}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c+j; zx) dx \\
&= \frac{\Gamma(c+j)\, F_1\left(b, c+j-b', b'; c; \dfrac{w}{\lambda}, \dfrac{w}{\lambda - z}\right)}{\lambda^{c+j-b'} (\lambda - z)^{b'}}, \\
&\quad (c+j > 0;\ \lambda > 0;\ c \neq 0, -1, \ldots;\ j = 0, \pm 1, \ldots;\ |w| + |z| < \lambda), \\
\mathrm{J}_c^{jj}(c+j, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) {}_1F_1(b'; c+j; zx) dx \\
&= \frac{\Gamma(c+j)(\lambda - w)^{b'-c-j}}{(\lambda - z - w)^{b'}} F_1\left(-j, c+j-b', b'; c; \frac{w}{w-\lambda}, \frac{w}{w+z-\lambda}\right), \\
&\quad (c+j > 0;\ \lambda > 0;\ c \neq 0, -1, \ldots;\ j = 0, \pm 1, \ldots;\ |w| + |z| < \lambda), \\
\mathrm{J}_c^{jj}(c+j, c; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) {}_1F_1(c; c+j; zx) dx \\
&= \frac{\Gamma(c+j)}{(\lambda - w)^j (\lambda - z - w)^c} F_1\left(-j, j, c; c; \frac{w}{w-\lambda}, \frac{w}{w+z-\lambda}\right), \\
&\quad (c+j > 0;\ \lambda > 0;\ c \neq 0, -1, \ldots;\ j = 0, \pm 1, \ldots;\ |w| + |z| < \lambda), \quad (5)
\end{aligned}$$

where F_1 is the first Appell function ([1], equation (1)). By mean of ([9], formula (8.3.5))

$$F_1(a; b, b'; c; w, z) = (1-w)^{-a} F_1\left(a, c-b-b', b'; c; \frac{w}{w-1}, \frac{z-w}{1-w}\right), \qquad (6)$$

it follows, for $j \geq p$,

$$\begin{aligned}
\mathrm{J}_c^{j(\pm p)}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c \pm p; zx) dx \\
&= \frac{\Gamma(c+j)}{\lambda^{c+j-b-b'} (\lambda - w)^b (\lambda - z)^{b'}} \sum_{k=0}^{j \mp p} \frac{(-j \pm p)_k (b')_k}{(c \pm p)_k\, k!} \left(1 - \frac{\lambda}{z}\right)^{-k} \\
&\times \sum_{r=0}^{j+k} \frac{(b)_r\, (-j-k)_r}{(c)_r\, r!} \left(1 - \frac{\lambda}{w}\right)^{-r} {}_2F_1\left(b+r, b'+k; c+r; \frac{w\, z}{(\lambda - z)(\lambda - w)}\right), \\
&\quad (c+j > 0;\ \lambda > 0;\ c, c \pm p \neq 0, -1, -2, \ldots;\ |w| + |z| < \lambda), \quad (7)
\end{aligned}$$

where for $p = j = 0, 1, 2, \ldots$

$$\begin{aligned}
\mathrm{J}_c^{jj}(b, b'; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(b'; c+j; zx) dx \\
&= \frac{\Gamma(c+j)}{\lambda^{c+j-b-b'}(\lambda-w)^b(\lambda-z)^{b'}} \sum_{r=0}^{j} \frac{(b)_r\,(-j)_r}{(c)_r\, r!} \left(1 - \frac{\lambda}{w}\right)^{-r} \\
&\times {}_2F_1\left(b+r, b'; c+r; \frac{w\,z}{(\lambda-z)(\lambda-w)}\right), \\
&(c+j>0;\ \lambda>0;\ c, c+j \neq 0, -1, -2, \ldots;\ |w|+|z|<\lambda). \qquad (8)
\end{aligned}$$

Setting $b' = c + j$, equation (4) yield

$$\begin{aligned}
\mathrm{J}_c^{j(\pm p)}(b, c+j; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(b; c; wx) {}_1F_1(c+j; c \pm p; zx) dx \\
&= \frac{\Gamma(c+j)}{(\lambda-z)^{c+j-b}(\lambda-z-w)^b} \sum_{k=0}^{j \mp p} \frac{(-j \pm p)_k (c+j)_k}{(c \pm p)_k\ k!} \left(1 - \frac{\lambda}{z}\right)^{-k} \\
&\times {}_2F_1\left(b, -j-k; c; \frac{w}{w+z-\lambda}\right),
\end{aligned}$$
$$(c+j>0;\ \lambda>0;\ c, c \pm p \neq 0, -1, -2, \ldots;\ p \geq 0;\ j = 0, \pm 1, \pm 2, \ldots;\ |w|+|z|<\lambda), \qquad (9)$$

By means of the Kummer's first transformation ${}_1F_1(b; c; z) = e^z {}_1F_1(c-b; c; -z)$, and the series representation

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m\, m!} x^m\ {}_2F_1(a+m, b'; c', y), \qquad (10)$$

it easily follows

$$\begin{aligned}
\mathrm{J}_c^{j(\pm p)}(c+j, b; \lambda, w, z) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} {}_1F_1(c+j; c; wx) {}_1F_1(b; c \pm p; zx)\, dx \\
&= \frac{\Gamma(c+j)}{(\lambda-w)^{c+j}} \sum_{k=0}^{j} \frac{(-j)_k (c+j)_k}{(c)_k\, k!} \left(1 - \frac{\lambda}{w}\right)^{-k} {}_2F_1\left(\begin{matrix} b, & c+j+k \\ & c \pm p \end{matrix}; \frac{z}{\lambda-w}\right), \\
&(c+j>0;\ \lambda>0;\ c, c \pm p \neq 0, -1, \ldots;\ |w|+|z|<\lambda). \qquad (11)
\end{aligned}$$

By means of the identity ([2], formula 5.14.3)

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-z)^k}{(b)_k} {}_1F_1(a; b+k; z) = {}_1F_1(a-n; b; z), \qquad (12)$$

it follows that

$$\mathrm{J}_c^{j(\pm p)}(c-j,b;\lambda,w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(c-j;c;wx){}_1F_1(b;c\pm p;zx)\,dx$$
$$=\frac{\Gamma(c+j)}{\lambda^{c+j}}\sum_{k=0}^{j}\frac{(-j)_k(c+j)_k}{(c)_k\,k!}\left(\frac{w}{\lambda}\right)^k F_2\left(\begin{matrix}c+j+k, & c, & b\\ & c+k & c\pm p\end{matrix};\frac{w}{\lambda},\frac{z}{\lambda}\right),$$
$$(c+j>0;\ \lambda>0;\ c,c\pm p\neq 0,-1,\ldots;\ |w|+|z|<\lambda). \tag{13}$$

By means of the identity ([2], formula 5.14.1)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(b-a)_k}{(b)_k}{}_1F_1(a;b+k;z)=\frac{(a)_n}{(b)_n}{}_1F_1(a+n;b+n;z), \tag{14}$$

it follows that

$$\mathrm{J}_{c+n}^{(j-n)(\pm p-n)}(b+n,b';\lambda,w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b+n;c+n;wx){}_1F_1(b';c\pm p;z\,x)\,dx$$
$$=\frac{\Gamma(c+j)}{\lambda^{c+j}}\frac{(c)_n}{(b)_n}\sum_{k=0}^{n}\frac{(-n)_k(c-b)_k}{(c)_k\,k!}F_2\left(\begin{matrix}c+j; & b, & b'\\ & c+k, & c\pm p\end{matrix};\frac{w}{\lambda},\frac{z}{\lambda}\right),$$
$$(c+j>0;\ \lambda>0; n=0,1,\ldots;\ c+n,c\pm p\neq 0,-1,\ldots;\ |w|+|z|<\lambda). \tag{15}$$

By means of the identity ([2], formula 5.14.5)

$${}_1F_1(b+n;c+n;w)=\frac{(c-1)_n(c)_n}{(b)_n(-w)^n}\sum_{k=0}^{n}\frac{(-n)_k(1-c)_k}{(2-c-n)_k\,k!}{}_1F_1(b,c-k,w) \tag{16}$$

it follows that

$$\mathrm{J}_{c+n}^{(j-n)(\pm p-n)}(b+n,b';\lambda,w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b+n;c+n;wx){}_1F_1(b';c\pm p;z\,x)\,dx$$
$$=\frac{(c-1)_n(c)_n}{(-w)^n\,(b)_n}\frac{\Gamma(c+j-n)}{\lambda^{c+j-n}}\sum_{k=0}^{n}\frac{(-n)_k(1-c)_k}{(2-c-n)_k\,k!}F_2\left(\begin{matrix}c+j-n; & b, & b'\\ & c-k, & c\pm p\end{matrix};\frac{w}{\lambda},\frac{z}{\lambda}\right),$$
$$(c\neq 0,\pm 1;\ldots,c+j>0;\ \lambda>0; n=0,1,\ldots;\ c+n,c\pm p\neq 0,-1,\ldots;\ |w|+|z|<\lambda). \tag{17}$$

By means of the identity ([2], formula 5.14.6)

$${}_1F_1(b+n;c;w)=\frac{(b-c+1)_n}{(b)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-c)_k}{(b-c+1)_k}{}_1F_1(b,c-k,w) \tag{18}$$

it follows

$$\mathrm{J}_c^{j(\pm p)}(b+n,b';\lambda,w,z) = \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b+n;c;wx){}_1F_1(b';c\pm p;z\,x)\,dx$$
$$= \frac{(b-c+1)_n}{(b)_n}\frac{\Gamma(c+j)}{\lambda^{c+j}}\sum_{k=0}^{n}\frac{(-n)_k(1-c)_k}{(b-c+1)_k\,k!}$$
$$\times F_2\begin{pmatrix} c+j; & b, & b' & ; \frac{w}{\lambda},\frac{z}{\lambda} \\ & c-k, & c\pm p & \end{pmatrix}$$
$$(c\neq 0,\pm1,\ldots;c+j>0;\ \lambda>0;n=0,1,\ldots;\ c,c\pm p\neq 0,-1,\ldots;\ |w|+|z|<\lambda). \tag{19}$$

By means of the identity ([2], formula 5.14.7)

$${}_1F_1(b-n;c-n;w) = \frac{(w)^n}{(1-c)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(1-c)_k}{w^k}{}_1F_1(b,c-k,w) \tag{20}$$

it follows

$$\mathrm{J}_{c-n}^{(j+n)(\pm p+n)}(b-n,b';\lambda,w,z) = \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b-n;c-n;wx){}_1F_1(b';c\pm p;z\,x)\,dx$$
$$= \frac{w^n\Gamma(c+j+n)}{\lambda^{c+j+n}(1-c)_n}\sum_{k=0}^{n}\frac{(-n)_k(1-c)_k}{k!\,(1-c-j-n)_k}\left(\frac{\lambda}{w}\right)^k$$
$$\times F_2\begin{pmatrix} c+j-k+n; & b, & b' & ; \frac{w}{\lambda},\frac{z}{\lambda} \\ & c-k, & c\pm p & \end{pmatrix}$$
$$(c\neq 0,\pm1,\ldots;c+j>0;\ \lambda>0;n=0,1,\ldots;\ c-n,c\pm p\neq 0,-1,\ldots;\ |w|+|z|<\lambda). \tag{21}$$

Since ([7], formula 7.2.4.68)

$$F_2(a;b,b;c,c;z,-z) = {}_4F_3\left(\frac{a}{2},\frac{a+1}{2},b,c-b;\frac{c}{2},\frac{c+2}{2},c;z^2\right). \tag{22}$$

it follows that

$$\mathrm{J}_c^{j0}(b,b;\lambda,w,-w) = \int_0^\infty x^{c+j-1}\,e^{-\lambda x}\,{}_1F_1(b;c;wx){}_1F_1(b;c;-wx)\,dx$$
$$= \frac{\Gamma(c+j)}{\lambda^{c+j}}{}_4F_3\begin{pmatrix} b, & c-b, & \frac{c+j}{2}, & \frac{c+j+1}{2} & ; \frac{w^2}{\lambda^2} \\ c, & \frac{c}{2}, & \frac{c+1}{2} & & \end{pmatrix},$$
$$(c+j>0;\ \lambda>0;\ c\neq 0,-1,-2,\ldots,\ |w|<\lambda), \tag{23}$$

$$\mathrm{J}_c^{10}(b,b;\lambda;w,-w)=\int_0^\infty x^c\,e^{-\lambda x}\,{}_1F_1(b;c;w\,x)\,{}_1F_1(b;c;-w\,x)\,dx$$
$$=\frac{\Gamma(c+1)}{\lambda^{c+1}}{}_3F_2\left(\begin{matrix}b, & c-b, & \frac{c}{2}+1 \\ c, & \frac{c}{2}, & \end{matrix};\frac{w^2}{\lambda^2}\right),$$
$$(c>-1;\ \lambda>0;\ |w|<\lambda), \tag{24}$$

$$\mathrm{J}_c^{10}\left(\frac{c}{2},\frac{c}{2};\lambda;w,-w\right)=\int_0^\infty x^c\,e^{-\lambda x}\,{}_1F_1\left(\frac{c}{2};c;w\,x\right)\,{}_1F_1\left(\frac{c}{2};c;-w\,x\right)\,dx$$
$$=\frac{\Gamma(c+1)}{\lambda^{c+1}}{}_2F_1\left(\begin{matrix}\frac{c}{2}, & \frac{c}{2}+1 \\ c, & \end{matrix};\frac{w^2}{\lambda^2}\right),$$
$$(c>-1;\ \lambda>0;\ |w|<\lambda). \tag{25}$$

On other hand, by means of ([7], formula 7.2.4.68)

$$F_2(a;b,c-b;c,c;z,z)=(1-z)^{-a}{}_4F_3\left(\frac{a}{2},\ \frac{a+1}{2},\ b,\ c-b;\ \frac{c}{2},\ \frac{c+2}{2},\ c;\frac{z^2}{(1-z)^2}\right), \tag{26}$$

it follows that

$$\mathrm{J}_c^{j0}(b,c-b;\lambda,z,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;z\,x){}_1F_1(c-b;c;z\,x)\,dx$$
$$=\frac{\Gamma(c+j)}{(\lambda-z)^{c+j}}{}_4F_3\left(\begin{matrix}b, & c-b, & \frac{c+j}{2}, & \frac{c+j+1}{2} \\ c, & \frac{c}{2}, & \frac{c+1}{2} & \end{matrix};\frac{z^2}{(\lambda-z)^2}\right),$$
$$(c+j>0;\lambda>0;\ c\neq 0,-1,-2,\ldots,\ |w|<\lambda), \tag{27}$$

$$\mathrm{J}_c^{10}(b,c-b;\lambda,z,z)=\int_0^\infty x^c\,e^{-\lambda x}\,{}_1F_1(b;c;z\,x)\,{}_1F_1(c-b;c;z\,x)\,dx$$
$$=\frac{\Gamma(c+1)}{(\lambda-z)^{c+1}}{}_3F_2\left(\begin{matrix}b, & c-b, & \frac{c}{2}+1 \\ c, & \frac{c}{2}, & \end{matrix};\frac{z^2}{(\lambda-z)^2}\right),$$
$$(c>-1;\ \lambda>0;\ |z|<|\lambda|), \tag{28}$$

$$\mathrm{J}_c^{10}\left(\frac{c}{2},\frac{c}{2};\lambda,z,z\right)=\int\limits_0^\infty x^c\,e^{-\lambda x}\left[{}_1F_1\left(\frac{c}{2};c;z\,x\right)\right]^2dx$$
$$=\frac{\Gamma(c+1)}{(\lambda-z)^{c+1}}{}_2F_1\left(\begin{matrix}\frac{c}{2}, & \frac{c}{2}+1\\ c, & \end{matrix};\frac{z^2}{(\lambda-z)^2}\right),$$
$$(c>-1;\;\lambda>0;\;|z|<\lambda). \tag{29}$$

By means of the identity ([6], Theorem 3, formula 29)

$$F_2(\sigma;\alpha_1,\alpha_2;\beta_1,\beta_2+n;w,z)=\frac{(\beta_2)_n}{(\beta_2-\alpha_2)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\alpha_2)_k}{(\beta_2)_k}$$
$$\times F_2(\sigma;\alpha_1,\alpha_2+k;\beta_1,\beta_2+k;w,z)$$
$$(|w|+|y|<1;n=0,1,2,\ldots;\beta_1,\beta_2\neq 0,-1,-2,\ldots;\beta_2>\alpha_2), \tag{30}$$

it easily follow

$$\mathrm{J}_c^{jp}(b,b';\lambda;w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(b';c+p;zx)dx$$
$$=\frac{\Gamma(c+j)\,(c)_p}{\lambda^{c+j}(c-b')_p}\sum_{k=0}^{p}\frac{(-p)_k\,(b')_k}{k!\,(c)_k}F_2\left(c+j;b,b'+k;c,c+k;\frac{w}{\lambda},\frac{z}{\lambda}\right),$$
$$(c+j>0;p\geq 0;\lambda>0;;\;c\neq 0,-1,-2,\ldots;$$
$$if\quad c-b'(negative\ integer),b'-c\geq p\ |w|+|z|<\lambda), \tag{31}$$

Further by means of ([6], Theorem 3, formula 29)

$$F_2\left(\begin{matrix}\sigma; & \alpha_1, & \alpha_2\\ & \beta_1, & \beta_2-n\end{matrix};w,z\right)=\frac{1}{\left[\prod\limits_{i=0}^{n}(\beta_2-i)\right]}\sum_{k=0}^{n}\binom{n}{k}\left[\prod_{j=0}^{n-k}(\beta_2-j)\right]\frac{(\sigma)_k(\alpha_2)_k}{(\beta_2)_k}z^k$$
$$\times F_2\left(\begin{matrix}\sigma+k; & \alpha_1, & \alpha_2+k\\ & \beta_1, & \beta_2+k\end{matrix};w,z\right),$$
$$(\beta_2\neq i,i=0,1,\ldots,n;\beta_1,\beta_2-n\neq 0,-1,-2,\ldots;n=0,1,2,\ldots;|w|+|z|<1), \tag{32}$$

it follows

$$\mathrm{J}_c^{j(-p)}(b,b';\lambda;w,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(b';c-p;zx)dx$$
$$=\frac{\Gamma(c+j)}{\lambda^{c+j}}\sum_{k=0}^{p}\frac{(-p)_k\,(b')_k\,(c+j)_k}{(c)_k\,(c-p)_k\,k!}\left(-\frac{z}{\lambda}\right)^kF_2\left(\begin{matrix}c+k+j; & b, & b'+k\\ & c, & c+k\end{matrix};\frac{w}{\lambda},\frac{z}{\lambda}\right),$$
$$(c+j>0;\;p\geq 0;\lambda>0;\;c\neq 0,-1,-2,\ldots;\;c-p\neq 0,-1,-2,\ldots;\;|w|+|z|<\lambda) \tag{33}$$

whence

$$\begin{aligned}
\mathrm{J}_c^{j(-p)}(b,c;\lambda;w,z) &= \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(c;c-p;zx)dx\\
&= \frac{\Gamma(c+j)(\lambda-z)^{b-c-j}}{(\lambda-w-z)^b}\sum_{k=0}^{p}\frac{(-p)_k(c+j)_k}{(c-p)_k\,k!}\left(\frac{z}{z-\lambda}\right)^k\\
&\times {}_2F_1\left(\begin{matrix}-k-j, & b\\ & c\end{matrix};\frac{w}{w+z-\lambda}\right)
\end{aligned}$$

$$(c+j>0;\ p\geq 0;\lambda>0;\ |w|+|z|<\lambda;\ c\neq 0,-1,-2,\dots;\ c-p\neq 0,-1,-2,\dots). \tag{34}$$

The following identities are straightforward consequences of the pervious integrals:

$$\begin{aligned}
\mathrm{J}_c^{j0}(c+j,0;\lambda,w,0) &= \int_0^\infty x^{c+j-1}\,e^{-\lambda\,x}{}_1F_1(c+j;\ c;\ w\,x)\,dx\\
&= \frac{\Gamma(c+j)}{(\lambda-w)^{c+j}}{}_2F_1\left(\begin{matrix}-j, & c+j\\ & c\end{matrix};\frac{w}{w-\lambda}\right),\\
&(c+j>0;\ \lambda>0;\ c\neq 0,-1,-2,\dots,\ |w|<\lambda),
\end{aligned} \tag{35}$$

$$\begin{aligned}
\mathrm{J}_c^{j(\pm p)}(0,b;\lambda,0,z) &= \int_0^\infty x^{c+j-1}\,e^{-\lambda x}\,{}_1F_1(b;c\pm p;z\,x)\,dx\\
&= \frac{\Gamma(c+j)}{\lambda^{c+j}}{}_2F_1\left(\begin{matrix}c+j, & b\\ & c\pm p\end{matrix};\frac{z}{\lambda}\right) = \frac{\Gamma(c+j)}{\lambda^{\pm p-b+c}(\lambda-z)^{b\mp p+j}}\\
&\times {}_2F_1\left(\begin{matrix}\pm p-j, & c\pm p-b\\ & c\pm p\end{matrix};\frac{z}{\lambda}\right),
\end{aligned}$$

$$(c+j>0;\ \lambda>0;\ c,c\pm p\neq 0,-1,-2,\dots;j=0,\pm 1,\dots;\ p=0,1,2\dots;\ |z|<\lambda), \tag{36}$$

$$\begin{aligned}
\mathrm{J}_c^{jj}(0,b;\lambda,0,z) &= \int_0^\infty x^{c+j-1}\,e^{-\lambda x}\,{}_1F_1(b;c+j;z\,x)\,dx = \frac{\Gamma(c+j)}{\lambda^{c-b+j}(\lambda-z)^b},\\
&(c+j>0;\lambda>0;j=0,\pm 1,\dots;\ |z|<\lambda),
\end{aligned} \tag{37}$$

$$\begin{aligned}
\mathrm{J}_c^{j0}(0,b;\lambda,0,z) &= \int_0^\infty x^{c+j-1}\,e^{-\lambda x}\,{}_1F_1(b;c;zx)\,dx\\
&= \frac{\Gamma(c+j)}{\lambda^{c+j-b}(\lambda-x)^b}\left[1+\frac{b\,z}{c\,(\lambda-z)}\sum_{k=1}^{j}{}_2F_1\left(\begin{matrix}-j+k, & b+1\\ & c+1\end{matrix};\frac{z}{z-\lambda}\right)\right],\\
&(c+j>0;\ \lambda>0;\ c\neq 0,-1,-2,\dots;\ |z|<\lambda).
\end{aligned} \tag{38}$$

$$\mathrm{J}_c^{00}(0,b;\lambda,0,z)=\int_0^\infty x^{c-1}\,e^{-\lambda x}\,{}_1F_1(b;c;zx)\,dx=\frac{\lambda^{b-c}\,\Gamma(c)}{(\lambda-z)^b},\ (c>0;\ \lambda>0;\ |z|<\lambda). \tag{39}$$

3 Gordon's integral and confluent hypergeometric polynomials

These are the main results of the paper.
In the case of $\alpha=-n$, the confluent hypergeometric function ${}_1F_1(\alpha;\beta;z)$ reduces to n-degree polynomial in z, namely ${}_1F_1(-n;\beta;z)=\sum_{k=0}^n(-n)_k\,z^k/((\beta)_k\,k!),\ n=0,1,\ldots$.. Thus,

$$\begin{aligned}\mathrm{J}_c^{j(\pm p)}(b,-n;\lambda,w,z)&=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(-n;c\pm p;zx)dx\\&=\frac{\Gamma(c+j)}{\lambda^{c+j-b}(\lambda-w)^b}\sum_{k=0}^n\frac{(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k{}_2F_1(-j-k,b;c;\frac{w}{w-\lambda}),\\&(c+j>0;\ \lambda>0;\ c,\ c\pm p\neq 0,-1,\ldots;p=0,1,\ldots;\ |w|<|\lambda|),\end{aligned} \tag{40}$$

where a direct differentiation of both sides with respect to z yields

$$\begin{aligned}\mathrm{J}_c^{(j+m)(\pm p+m)}(b,m-n;\lambda,w,z)&=\int_0^\infty x^{c+j+m-1}e^{-\lambda x}{}_1F_1(b;c;wx){}_1F_1(m-n;c\pm p+m;zx)\,dx\\&=\frac{(-1)^m\Gamma(c+j)(c\pm p)_m}{(-n)_m z^m\lambda^{c+j-b}(\lambda-w)^b}\sum_{k=m}^n\frac{(-k)_m(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k\\&\quad\times{}_2F_1(-j-k,b;c;\frac{w}{w-\lambda}),\\&(m\leq n;c+j+m>0;\ \lambda>0;\ c,c,\ c\pm p+m\neq 0,-1,\ldots;\ |w|<|\lambda|).\end{aligned} \tag{41}$$

Further, setting $b=-m$ in equation (40) implies

$$\begin{aligned}\mathrm{J}_c^{jp}(-m,-n;\lambda;w,z)&=\int_0^\infty x^{c+j-1}e^{-\lambda\,x}{}_1F_1(-m;c;w\,x)\,{}_1F_1(-n;c\pm p;z\,x)\,dx\\&=\frac{\Gamma(c+j)}{\lambda^{c+j}}\sum_{k=0}^m\frac{(c+j)_k(-m)_k}{(c)_k\,k!}\left(\frac{w}{\lambda}\right)^k{}_2F_1(-n,c+j+k;c\pm p;\frac{z}{\lambda})\\&\equiv\frac{\Gamma(c+j)}{\lambda^{c+j}}\sum_{k=0}^n\frac{(c+j)_k(-n)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k{}_2F_1(-m,c+j+k;c;\frac{w}{\lambda}),\\&(c+j>0,\ \lambda>0;\ c,c\pm p\neq 0,-1,\ldots;j=0,\pm 1,\ldots;n,m=0,1,\ldots),\end{aligned} \tag{42}$$

where a direct differentiation of both sides with respect to w yields

$$\int_0^\infty x^{c+j+l-1}e^{-\lambda x}{}_1F_1(l-m;c+l;w\,x)\,{}_1F_1(-n;c\pm p;z\,x)\,dx$$
$$=\frac{(-1)^l\Gamma(c+j)(c)_l}{\lambda^{c+j}w^l(-m)_l}\sum_{k=l}^{m}\frac{(c+j)_k(-m)_k(-k)_l}{(c)_k\,k!}\Big(\frac{w}{\lambda}\Big)^k{}_2F_1(-n,c+j+k;c\pm p;\frac{z}{\lambda}),$$
$$(l\le m;c+j+l>0,\ \lambda>0;\ c+l,c\pm p\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots),\tag{43}$$

with a further differentiation of both sides with respect to z yields

$$\int_0^\infty x^{c+j+k+s-1}e^{-\lambda x}{}_1F_1(l-m;c+l;w\,x)\,{}_1F_1(s-n;c\pm p+s;z\,x)\,dx$$
$$=\frac{(-1)^l\Gamma(c+j)(c)_l}{\lambda^{c+j+s}w^l(-m)_l}\sum_{k=l}^{m}\frac{(c+j+k)_s(c+j)_k(-m)_k(-k)_l}{(c)_k\,k!}\Big(\frac{w}{\lambda}\Big)^k$$
$$\times\,{}_2F_1(s-n,c+j+k+s;c\pm p+s;\frac{z}{\lambda})$$
$$(s\le n;l\le m;c+j+l+s>0,\ \lambda>0;\ c+l,c\pm p+s\neq 0,-1,\dots;$$
$$s,l,j=0,\pm1,\dots;n,m=0,1,\dots).\tag{44}$$

If $z=\lambda$, equation (40) reads

$$\mathrm{J}_c^{j(\pm p)}(-m,-n;\lambda;w,\lambda)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-m;c;w\,x)\,{}_1F_1(-n;c\pm p;\lambda\,x)\,dx$$
$$=\frac{\Gamma(c+j)(\pm p-j)_n}{\lambda^{c+j}(\pm p+c)_n}{}_3F_2(-m,c+j,1+j\mp p;c,1+j-n\mp p;\frac{w}{\lambda}),$$
$$(c+j>0,\ \lambda>0;\ c,c\pm p\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots;1+j-n\pm p\neq 0,-1,\dots),\tag{45}$$

and if $w=\lambda$ equation (40) reads

$$\mathrm{J}_c^{j(\pm p)}(-m,-n;\lambda;\lambda,z)=\int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-m;c;\lambda\,x)\,{}_1F_1(-n;c\pm p;z\,x)\,dx$$
$$=\frac{\Gamma(c+j)(-j)_m}{\lambda^{c+j}(c)_m}{}_3F_2(-n,c+j,1+j;c\pm p,1+j-m;\frac{z}{\lambda}),$$
$$(c+j>0,\ \lambda>0;\ c,c\pm p\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots;1+j-m\neq 0,-1,\dots).\tag{46}$$

Further if $w = \lambda$, equation (45) reads

$$
\begin{aligned}
\mathrm{J}_c^{jp}(-m,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x}\, {}_1F_1(-m;c;\lambda x)\, {}_1F_1(-n;c\pm p;\lambda x)\, dx \\
&= \frac{\Gamma(c+j)\,(\pm p-j)_n}{\lambda^{c+j}\,(\pm p+c)_n}\, {}_3F_2(-m,c+j,1+j\mp p;c,1+j-n\mp p;1) \\
&= \frac{\Gamma(c+j)(-j)_m}{\lambda^{c+j}(c)_m} {}_3F_2(-n,c+j,1+j;c\pm p,1+j-m;1),
\end{aligned}
$$

$$
(c+j>0,\ \lambda>0;\ c,c\pm p\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots;1+j-n\pm p,\ 1+j-m\neq 0,-1,\dots), \tag{47}
$$

whenece

$$
\begin{aligned}
\mathrm{J}_c^{j0}(-m,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x}\, {}_1F_1(-m;c;\lambda x)\, {}_1F_1(-n;c;\lambda x)\, dx \\
&= \frac{\Gamma(c+j)\,(-j)_n}{\lambda^{c+j}\,(c)_n}\, {}_3F_2(-m,c+j,1+j;c,1+j-n;1), \\
&\equiv \frac{\Gamma(c+j)\,(-j)_m}{\lambda^{c+j}\,(c)_m}\, {}_3F_2(-n,c+j,1+j;c,1+j-m;1),
\end{aligned}
$$

$$
(c+j>0,\ \lambda>0;\ c\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots;1+j-n,\ 1+j-m\neq 0,-1,\dots). \tag{48}
$$

If $j=0$, equation (47)

$$
\begin{aligned}
\mathrm{J}_c^{0p_\pm}(-n,-m;\lambda,\lambda,\lambda) &= \int_0^\infty x^{c-1} e^{-\lambda x}\, {}_1F_1(-n;c;\lambda x) {}_1F_1(-m;c\pm p;\lambda x)\, dx \\
&= \frac{\Gamma(c)}{\lambda^c}\frac{m!}{(m-n)!}\frac{(\pm p)_{m-n}}{(c\pm p)_m},
\end{aligned}
$$

$$
(m\geq n;\ c>0,\ \lambda>0;\ c\pm p\neq 0,-1,\dots;j=0,\pm1,\dots;n,m=0,1,\dots). \tag{49}
$$

If $m=n$, equation (47)

$$
\begin{aligned}
\mathrm{J}_c^{j0}(-n,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} \left[{}_1F_1(-n;c;\lambda x)\right]^2 dx \\
&= \frac{\Gamma(c+j)\,(-j)_n}{\lambda^{c+j}\,(c)_n}\, {}_3F_2(-n,c+j,1+j;c,1+j-n;1),
\end{aligned}
$$

$$
(c+j>0,\ \lambda>0;\ c\neq 0,-1,\dots;j=0,\pm1,\dots;n=0,1,\dots;1+j-n\neq 0,-1,\dots). \tag{50}
$$

The condition $1+j-n\neq 0,-1,-2,\dots$ in (50) can be softened using the identity

$$
(-j)_n\, {}_3F_2(-n,c+j,1+j;c,1+j-n;1) = n!\, {}_3F_2(-n,-j,j+1;c,1;1), \tag{51}
$$

to yield

$$\begin{aligned} \mathrm{J}_c^{j0}(-n,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c+j-1} e^{-\lambda x} \left[{}_1F_1(-n;c;\lambda x)\right]^2 \, dx \\ &= \frac{\Gamma(c+j)\, n!}{\lambda^{c+j}\,(c)_n}\, {}_3F_2(-n,-j,1+j;c,1;1), \end{aligned}$$

$$(c+j>0;\ \lambda>0;\ c\neq 0,-1,-2,\ldots;\, j=0,\pm1,\pm2,\ldots;\, n=0,1,2,\ldots). \tag{52}$$

and thus

$$\begin{aligned} \mathrm{J}_c^{10}(-n,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c}\, e^{-\lambda x} \left[{}_1F_1(-n;c;\lambda x)\right]^2 \, dx \\ &= \frac{\Gamma(c)\, n!}{\lambda^{c+1}\,(c)_n}\,(c+2n)\,, \quad (c>0;\ \lambda>0). \end{aligned} \tag{53}$$

From the symmetric property $j \longleftrightarrow -j-1$ of ${}_3F_2(-n,-j,1+j;c,1;1)$, it also follow

$$\begin{aligned} \mathrm{J}_c^{(-j-1)0}(-n,-n;\lambda;\lambda,\lambda) &= \int_0^\infty x^{c-j-2} e^{-\lambda x} \left[{}_1F_1(-n;c;\lambda x)\right]^2 \, dx \\ &= \frac{\Gamma(c-j-1)\, n!}{\lambda^{c-j-1}\,(c)_n}\, {}_3F_2(-n,-j,1+j;c,1;1), \end{aligned}$$

$$(c-j-2>0;\ \lambda>0;\ c\neq 0,-1,-2,\ldots;\, j=0,\pm1,\pm2,\ldots;\, n=0,1,2,\ldots) \tag{54}$$

whence

$$\mathrm{J}_c^{(-j-1)0}(-n,-n;\lambda;\lambda,\lambda) = \frac{\Gamma(c-j-1)\lambda^{2j+1}}{\Gamma(c+j)}\, \mathrm{J}_c^{j0}(-n,-n;\lambda;\lambda,\lambda), \tag{55}$$

for example

$$\mathrm{J}_c^{(-2)0}(-n,-n;\lambda;\lambda,\lambda) = \frac{\Gamma(c-2)\lambda^3}{\Gamma(c+1)}\, \mathrm{J}_c^{10}(-n,-n;\lambda;\lambda,\lambda) = \frac{\Gamma(c-2)n!}{c\lambda^{c-2}\,(c)_n}\,(c+2n)\,. \tag{56}$$

Further recurrence relations of this type are developed in the appendix. Note, from equations (42) and (45), it follows

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-n)_k\,(c+j)_k}{(c\pm p)_k\, k!}\, {}_2F_1\left(-m, c+j+k; c; \frac{w}{\lambda}\right) &= \frac{(\pm p-j)_n}{(\pm p+c)_n} \\ &\times {}_3F_2\left(-m, c+j, 1+j\mp p; c, 1+j-n\mp p; \frac{w}{\lambda}\right). \end{aligned} \tag{57}$$

An important class of W. Gordon's integral occur in the case of $w = k_1$, $z = k_2$, and $\lambda = (k_1+k_2)/2$, namely,

$$\begin{aligned}
\mathrm{J}_c^{j(\pm p)}\left(-n,-m;\frac{k_1+k_2}{2},k_1,k_2\right) &= \int_0^\infty x^{c+j-1}e^{-(k_1+k_2)x/2}{}_1F_1(-n;c;k_1\,x){}_1F_1(-m;c\pm p;k_2\,x)dx\\
&= \frac{2^{c+j}\Gamma(c+j)}{(k_1+k_2)^{c+j}}\,F_2\left(\begin{matrix}c+j; & -n, & -m\\ & c, & c\pm p\end{matrix};\frac{2k_1}{k_1+k_2},\frac{2k_2}{k_1+k_2}\right),\\
&(k_1+k_2>0;c+j>0;c,c\pm p\neq 0,\pm1,\dots)
\end{aligned} \tag{58}$$

equivalently,

$$\begin{aligned}
&\mathrm{J}_c^{j(\pm p)}\left(-n,-m;\frac{k_1+k_2}{2},k_1,k_2\right) = \int_0^\infty x^{c+j-1}e^{-(k_1+k_2)x/2}{}_1F_1(-n;c;k_1\,x){}_1F_1(-m;c\pm p;k_2\,x)dx\\
&= \begin{cases}
\dfrac{\Gamma(c+j)(\pm p-j)_m}{k_1^{c+j}(\pm p+c)_m}{}_3F_2(-n,c+j,1+j\mp p;c,1+j-m\mp p;1), & \text{if } k_1=k_2,\\[2ex]
\dfrac{2^{c+j}\Gamma(c+j)}{(k_1+k_2)^{c+j}}\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^m \sum_{i=0}^{\min\{j\mp p,m\}}\dfrac{(-j\pm p)_i(-m)_i}{(c\pm p)_i\;i!}\left(\dfrac{2k_2}{k_2-k_1}\right)^i & \\
\times F_1\left(-n,c+j+m,i-m;c;\dfrac{2k_1}{k_1+k_2},\dfrac{2k_1}{k_1-k_2}\right), & \text{if } k_1\neq k_2.
\end{cases}
\end{aligned} \tag{59}$$

and further equivalent to

$$\begin{aligned}
&\mathrm{J}_c^{j(\pm p)}\left(-n,-m;\frac{k_1+k_2}{2},k_1,k_2\right) = \int_0^\infty x^{c+j-1}e^{-(k_1+k_2)x/2}{}_1F_1(-n;c;k_1\,x){}_1F_1(-m;c\pm p;k_2\,x)dx\\
&= \begin{cases}
\dfrac{\Gamma(c+j)(\pm p-j)_m}{k_1^{c+j}(\pm p+c)_m}{}_3F_2(-n,c+j,1+j\mp p;c,1+j-m\mp p;1), \text{if } k_1=k_2,\\[2ex]
\dfrac{(-1)^n2^{c+j}\Gamma(c+j)}{(k_1+k_2)^{c+j}}\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^{m+n} \sum_{i=0}^{\min\{j\mp p,m\}}\dfrac{(-j\pm p)_i(-m)_i}{(c\pm p)_i\;i!}\left(\dfrac{2k_2}{k_2-k_1}\right)^i\\
\times\displaystyle\sum_{r=0}^{j+i}\dfrac{(-n)_r(-j-i)_r}{(c)_r\,r!}\left(\dfrac{2k_2}{k_2-k_1}\right)^r{}_2F_1\left(r-n,i-m;c+r;\dfrac{-4k_1k_2}{(k_1-k_2)^2}\right),\ \text{if } k_1\neq k_2.
\end{cases}
\end{aligned} \tag{60}$$

In particular

$$\mathrm{J}_c^{j0}\left(-n,-m;\frac{k_1+k_2}{2},k_1,k_2\right)=\int_0^\infty x^{c+j-1}e^{-(k_1+k_2)x/2}{}_1F_1(-n;c;k_1\,x){}_1F_1(-m;c;k_2\,x)dx$$
$$=\begin{cases}\dfrac{\Gamma(c+j)(-j)_m}{k_1^{c+j}(\pm p+c)_m}{}_3F_2(-n,c+j,1+j;c,1+j-m;1),, & (k_1=k_2, m\le j),\\[2ex] \dfrac{(-1)^n2^{c+j}\Gamma(c+j)}{(k_1+k_2)^{c+j}}\left(\dfrac{k_1-k_2}{k_1+k_2}\right)^{m+n}\sum_{i=0}^m\dfrac{(-j)_i(-m)_i}{(c)_i\ i!}\left(\dfrac{2k_2}{k_2-k_1}\right)^i & \\ \times\displaystyle\sum_{r=0}^{j+i}\dfrac{(-n)_r(-j-i)_r}{(c)_r\,r!}\left(\dfrac{2k_2}{k_2-k_1}\right)^r{}_2F_1\left(r-n,i-m;c+r;\dfrac{-4k_1k_2}{(k_1-k_2)^2}\right), & (k_1\neq k_2, m\le j)\end{cases}\tag{61}$$

and

$$\mathrm{J}_c^{00}\left(-m,-n;\frac{k_1+k_2}{2};k_1,k_2\right)=\int_0^\infty x^{c-1}e^{-(k_1+k_2)x/2}{}_1F_1(-m;c;k_1\,x)\,{}_1F_1(-n;c;k_2\,x)\,dx$$
$$=\frac{2^c\Gamma(c)}{(k_1+k_2)^c}\sum_{k=0}^m\frac{(-m)_k}{k!}\left(\frac{2k_1}{k_1+k_2}\right)^k{}_2F_1(-n,c+k;c;\frac{2k_2}{k_1+k_2}),$$
$$(c>0,\ \lambda>0;\ c\neq 0,-1,-2,\dots;n,m=0,1,2,\dots),\tag{62}$$

where, generally,

$$\mathrm{J}_c^{00}(-m,-n;\lambda;w,z)=\int_0^\infty x^{c-1}e^{-\lambda\,x}{}_1F_1(-m;c;w\,x)\,{}_1F_1(-n;c;z\,x)\,dx$$
$$=\frac{\Gamma(c)}{\lambda^c}\sum_{k=0}^m\frac{(-m)_k}{k!}\left(\frac{w}{\lambda}\right)^k{}_2F_1(-n,c+k;c;\frac{z}{\lambda}),$$
$$(c>0,\ \lambda>0;\ c\neq 0,-1,-2,\dots;n,m=0,1,2,\dots),\tag{63}$$

from which the classical orthogonality property of the confluent hypergeometric functions follows, namely,

$$\mathrm{J}_c^{00}(-m,-n;\lambda;\lambda,\lambda)=\int_0^\infty x^{c-1}e^{-\lambda\,x}{}_1F_1(-m;c;\lambda\,x)\,{}_1F_1(-n;c;\lambda\,x)\,dx=\frac{\Gamma(c)\,n!}{\lambda^c\,(c)_n}\delta_{nm},$$
$$(c>0;\ \lambda>0;\ c\neq 0,-1,-2,\dots;\delta_{nm}=0\ if\ n\neq m,\delta_{nm}=1\ if\ n=m),\tag{64}$$

using $\sum_{k=0}^m(-m)_k(-k)_n/k!=n!\delta_{nm}$. The same conclusion also follows from equation (48) using the fact that

$$\lim_{j\to 0}(-j)_m\,{}_3F_2(-n,c+j,1+j;c,1+j-m;1)=n!\,\delta_{nm}.$$

If in equation (1), $b = b' = -n$ and $p = 0$, it follows

$$\begin{aligned}\mathrm{J}_c^{j0}(-n,-n;\lambda;w,z) &= \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-n;c;w\,x)\,{}_1F_1(-n;c;z\,x)\,dx\\ &= \frac{n!\,\Gamma(c+j)}{\lambda^{c+j}(c)_n}\sum_{k=0}^{n}\frac{(c+j)_k(-n)_k}{(c)_k\,k!}\left(\frac{z}{\lambda}\right)^k P_n^{(c-1,j+k-n)}\left(1-\frac{2w}{\lambda}\right),\end{aligned} \tag{65}$$

where $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial of order α, β and degree n in z. The relation $P_n^{(a,b)}(-1) = (-1)^n(b+1)_n/n!$ reduce the equation (65) to

$$\begin{aligned}\mathrm{J}_c^{j0}(-n,-n;\lambda;\lambda,z) &= \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-n;c;\lambda\,x)\,{}_1F_1(-n;c;z\,x)\,dx\\ &= \frac{\Gamma(c+j)(-j)_n}{\lambda^{c+j}\,(c)_n}\,{}_3F_2\left(-n,c+j,1+j;c,1+j-n;\frac{z}{\lambda}\right),\\ &(c+j>0,\ \lambda>0;\ c\neq 0,-1,\dots;n=0,1,\dots;1+j-n\neq 0,-1,\dots),\end{aligned} \tag{66}$$

as expected. From equation (66), it follows

$$\begin{aligned}\mathrm{J}_c^{n0}(-n,-n;\lambda;\lambda,z) &= \int_0^\infty x^{c+n-1}e^{-\lambda x}{}_1F_1(-n;c;\lambda\,x)\,{}_1F_1(-n;c;z\,x)\,dx\\ &= \frac{(-1)^n\,\Gamma(c)\,n!}{\lambda^{c+n}}\,{}_3F_2\left(-n,c+n,1+n;c,1;\frac{z}{\lambda}\right),\\ &(c+n>0,\ \lambda>0;\ c\neq 0,-1,\dots).\end{aligned} \tag{67}$$

For $n \geq m$

$$\begin{aligned}\mathrm{J}_c^{(n-m)(\pm p)}(-n,-m;\lambda,\lambda,z) &= \int_0^\infty x^{c+n-m-1}e^{-\lambda x}{}_1F_1(-n;c;\lambda\,x){}_1F_1(-m;c\pm p;z\,x)\,dx\\ &= \frac{(-1)^{m+n}\,\Gamma(c)\,n!}{\lambda^{c+n-m}(c\pm p)_m}\left(\frac{z}{\lambda}\right)^m,\\ &(c+n-m>0;\lambda>0;c,c\pm p\neq 0,-1,\dots;p\geq 0).\end{aligned} \tag{68}$$

The following integral follows immediately

$$\begin{aligned}\mathrm{J}_c^{j(\pm p)}(0,-n;\lambda,0,z) &= \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-n;c\pm p;z\,x)dx\\ &= \frac{\Gamma(c+j)}{\lambda^{c+j}}{}_2F_1\left(-n,c+j;c\pm p;\frac{z}{\lambda}\right),\\ &(c+j>0;\ \lambda>0;\ c\pm p\neq 0,-1,\dots;p=0,1,\dots)\end{aligned} \tag{69}$$

whence

$$\mathrm{J}_c^{j(\pm p)}(0,-n;\lambda,0,\lambda) = \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-n;c\pm p;\lambda x)dx$$
$$= \begin{cases} \dfrac{\Gamma(c+j)}{\lambda^{c+j}}\dfrac{(\pm p-j)_n}{(c\pm p)_n}, & \text{if } j\mp p\ge n,\\ 0, & \text{if } j\mp p<n, \end{cases}$$
$$(c+j>0;\ \lambda>0;\ c\pm p\neq 0,-1,\dots;p=0,1,\dots), \quad (70)$$

and if $p=j$,

$$\mathrm{J}_c^{jj}(0,-n;\lambda,0,z) = \int_0^\infty x^{c+j-1}e^{-\lambda x}{}_1F_1(-n;c+j;z\,x)dx$$
$$= \frac{\Gamma(c+j)}{\lambda^{c+j}}\left(1-\frac{z}{\lambda}\right)^n, \quad (c+j>0;\lambda>0). \quad (71)$$

and

$$\mathrm{J}_c^{j0}(0,-n;\lambda,0,\lambda) = \int_0^\infty x^{c+n-1}e^{-\lambda x}{}_1F_1(-n;c;\lambda\,x)dx = \frac{(-1)^n\,n!\,\Gamma(c)}{\lambda^{c+n}},$$
$$(c+n>0;\ \lambda>0;\ c\neq 0,-1,\dots). \quad (72)$$

4 Gordon's Integral and special functions

The generalized Laguerre polynomials are defined, for integer n, in terms of confluent hypergeometric functions by

$$L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!}{}_1F_1(-n;\lambda+1;z), \quad (73)$$

thus,

$$\int_0^\infty x^{c+j-1}e^{-\lambda x}L_n^{c\pm p-1}(z\,x){}_1F_1(b;c;w\,x)dx$$
$$= \frac{\Gamma(c+j)(c\pm p)_n}{n!\lambda^{c+j}}\sum_{k=0}^n\frac{(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k{}_2F_1\left(c+j+k,b;c;\frac{w}{\lambda}\right)$$
$$(c+j>0;\ \lambda>0;\ c,\ c\pm p\neq 0,-1,\dots;p=0,1,\dots;\ |w|<|\lambda|), \quad (74)$$

whence, if $b=0$,

$$\int_0^\infty x^{c+j-1}e^{-\lambda x}L_n^{c\pm p-1}(z\,x)dx = \begin{cases} \dfrac{\Gamma(c+j)(c\pm p)_n}{n!\,\lambda^{c+j}}{}_2F_1\left(-n,c+j;c\pm p;\dfrac{z}{\lambda}\right), \\ (z\neq\lambda,\ c\pm p\neq 0,-1,\dots) \\ \\ \dfrac{\Gamma(c+j)(\pm p-j)_n}{\lambda^{c+j}\,n!}, (z=\lambda,\ j\mp p\ge n,) \end{cases}$$
$$(c+j>0;\ \lambda>0;\ p=0,1,\dots). \quad (75)$$

From equation (75), it follows

$$\int_0^\infty x^c\, e^{-\lambda x}\, L_n^c(\lambda x)\, dx = \begin{cases} \dfrac{\Gamma(c+1)}{\lambda^{1+c}}, & \text{if } n=0,\\ 0, & \text{if } n\geq 1; c>-1, \lambda>0,\ n=0,1,\dots, \end{cases}$$

$$\int_0^\infty x^{c+j}\, e^{-\lambda x}\, L_n^c(\lambda x)\, dx = \begin{cases} \dfrac{\Gamma(c+j+1)\,(-j)_n}{\lambda^{j+c+1}\, n!}, & \text{if } n<j,\\ (-1)^n\dfrac{\Gamma(c+n+1)}{\lambda^{n+c+1}}, & \text{if } n=j,\\ 0, & \text{if } n>j; c+j>-1,\ \lambda>0,\ n=0,1,\dots, \end{cases}$$

$$\int_0^\infty x^c\, e^{-\lambda x}\, L_n^{c-p}(\lambda x)\, dx = \begin{cases} \dfrac{\Gamma(c+1)(-p)_n}{\lambda^{c+1}\, n!}, & \text{if } n<p,\\ (-1)^n\dfrac{\Gamma(c+1)}{\lambda^{c+1}}, & \text{if } n=p,\\ 0, & \text{if } n>p; c>-1, \lambda>0, c-p\geq 0. \end{cases} \tag{76}$$

Since

$$\frac{d^m}{dz^m}L_n^\lambda(az) = (-a)^m L_{n-m}^{\lambda+m}(az)$$

it easily follows, for $n\geq m$ and $m=0,1,2,\dots$, that

$$\int_0^\infty x^{c+j+m-1}e^{-\lambda x}L_{n-m}^{c\pm p+m-1}(zx)\,{}_1F_1(b;c;wx)dx = \frac{(c\pm p)_n\Gamma(c+j)}{n!\,z^m\,\lambda^{c+j}}\sum_{k=m}^{n}\frac{(-k)_m(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k {}_2F_1(c+j+k,b;c;\frac{w}{\lambda}). \tag{77}$$

and, for $\mu=0,1,2,\dots,m\leq n$,

$$\int_0^\infty x^{c+j+m+\mu-1}e^{-\lambda x}L_{n-m}^{c\pm p+m-1}(zx)\,{}_1F_1(b+\mu;c+\mu;w\,x)dx = \frac{\Gamma(c+j)(c+p)_n}{n!\,z^m\,\lambda^{c+j+\mu}}\sum_{k=0}^{n}\frac{(-k)_m(-n)_k(c+j)_k(c+j+k)_\mu}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k \times {}_2F_1(c+j+k+\mu,b+\mu;c+\mu;\frac{w}{\lambda}). \tag{78}$$

On other hand,

$$\int_0^\infty x^{c+j-1}e^{-\lambda x}L_n^{c\pm p-1}(zx)L_m^{c-1}(w\,x)dx = \frac{(c)_m(c\pm p)_n\Gamma(c+j)}{m!\,n!\,\lambda^{c+j}} \times \sum_{k=0}^{n}\frac{(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k {}_2F_1\left(c+j+k,-m;c;\frac{w}{\lambda}\right),$$
$$(c+j>0;\ \lambda>0;\ c,\ c\pm p\neq 0,-1,\dots;p=0,1,\dots). \quad (79)$$

and by direct differentiation s-times, with respect to w, of both sides

$$\int_0^\infty x^{c+j+s-1}e^{-\lambda x}L_n^{c\pm p-1}(zx)L_{m-s}^{c+s-1}(w\,x)dx = \frac{(-1)^s(c)_m(c\pm p)_n\Gamma(c+j)}{m!\,n!\,\lambda^{c+j+s}}\sum_{k=0}^{n}\frac{(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\frac{(c+j+k)_s(-m)_s}{(c)_s}\left(\frac{z}{\lambda}\right)^k \times {}_2F_1\left(c+j+k+s,s-m;c+s;\frac{w}{\lambda}\right),$$
$$(m\geq s;c+j+s>0;\ \lambda>0;\ c+s,\ c\pm p\neq 0,-1,\dots;p=0,1,\dots). \quad (80)$$

and further differentiation of both sides μ-times, with respect to z,

$$\int_0^\infty x^{c+j+s+\mu-1}\,e^{-\lambda x}\,L_{n-\mu}^{c\pm p+\mu-1}(zx)\,L_{m-s}^{c+s-1}(w\,x)\,dx = \frac{(-m)_s(c)_m(c\pm p)_n\Gamma(c+j)}{(-1)^s\,m!\,n!\,z^\mu\,\lambda^{c+j+s}(c)_s}\sum_{k=\mu}^{n}\frac{(-k)_\mu(c+j+k)_s(-n)_k(c+j)_k}{(c\pm p)_k\,k!}\left(\frac{z}{\lambda}\right)^k \times {}_2F_1(c+j+k+s,s-m;c+s;\frac{w}{\lambda}),$$
$$(s\leq m;\ \mu\leq n;\ c+j+s+\mu>0;\ \lambda>0;\ c,\ c\pm p\neq 0,-1,\dots;p=0,1,\dots). \quad (81)$$

If $w=\lambda$, equation (79) reads

$$\int_0^\infty x^{c+j-1}e^{-\lambda x}L_n^{c\pm p-1}(zx)L_m^{c-1}(\lambda\,x)dx = \frac{(-j)_m(c\pm p)_n\Gamma(c+j)}{m!\,n!\,\lambda^{c+j}} \times {}_3F_2\left(-n,c+j,1+j;c\pm p,1+j-m;\frac{z}{\lambda}\right),$$
$$(j\geq m;\ c+j>0;\ \lambda>0;\ c,\ c\pm p\neq 0,-1,\dots;p=0,1,\dots). \quad (82)$$

and if $p=0$ it yields

$$\int_0^\infty x^{c+j-1}e^{-\lambda x}L_n^{c+j-1}(zx)L_m^{c-1}(\lambda\,x)dx = \frac{(-j)_m\Gamma(c+j+n)}{m!\,n!\,\lambda^{c+j}}{}_2F_1\left(-n,1+j;1+j-m;\frac{z}{\lambda}\right),$$
$$(j\geq m;\ c+j>0;\ \lambda>0). \quad (83)$$

and by taken limit of both sides as $j \to 0$

$$\int_0^\infty x^{c-1} e^{-\lambda x} L_n^{c-1}(zx) L_m^{c-1}(\lambda\, x) dx = \frac{(-1)^m\, z^m \Gamma(c+n)(-n)_m}{m!\, n!\, \lambda^{c+n}\, (\lambda - z)^{m-n}}, \quad (n \geq m;\ c > 0; \lambda > 0; |z| < \lambda), \tag{84}$$

thus

$$\int_0^\infty x^{c-1} e^{-\lambda x} L_n^{c-1}(\lambda x) L_m^{c-1}(\lambda\, x) dx = \frac{(c)_n \Gamma(c)}{m!\, \lambda^c} \delta_{m,n}, \quad (c > 0;\ \lambda > 0; n, m = 0, 1, \dots). \tag{85}$$

By means of $H_{2n}(\sqrt{z}) = (-1)^n\,(2n)!\,{}_1F_1(-n; 0.5; z)/n!$, it follows using (74) that

$$\int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x}\, L_n^{\pm p-\frac{1}{2}}(zx)\, H_{2n}(\sqrt{wx})\, dx = \frac{(-1)^n (2n)! (\pm p + \frac{1}{2})_n \Gamma(j + \frac{1}{2})}{(n!)^2 \lambda^{j+\frac{1}{2}}} \times \sum_{k=0}^{n} \frac{(-n)_k (j+\frac{1}{2})_k}{(\pm p + \frac{1}{2})_k\, k!} \left(\frac{z}{\lambda}\right)^k {}_2F_1\left(j + k + \frac{1}{2}, -n; \frac{1}{2}; \frac{w}{\lambda}\right), \quad (j > 1/2; p, n = 0, 1, \dots), \tag{86}$$

from which it follows

$$\int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x}\, L_n^{\pm p-\frac{1}{2}}(zx)\, H_{2n}(\sqrt{\lambda x})\, dx = \frac{(-1)^n (2n)! (-j)_n (\pm p + \frac{1}{2})_n \Gamma(j + \frac{1}{2})}{(n!)^2\, (\frac{1}{2})_n\, \lambda^{j+\frac{1}{2}}} \times {}_3F_2\left(j + \frac{1}{2}, 1 + j, -n; 1 + j - n, \pm p + \frac{1}{2}; \frac{z}{\lambda}\right), \quad (j > 1/2; \lambda > 0;\quad p, n = 0, 1, \dots) \tag{87}$$

However, by means of

$$\lim_{j \to 0} (-j)_n\, {}_3F_2\left(-n, \frac{1}{2} + j, j + 1; \pm p + \frac{1}{2}, j + 1 - n; 1\right) = \frac{(2n)!}{4^n (\pm p + \frac{1}{2})_n} \tag{88}$$

it easily follows that

$$\int_0^\infty x^{-\frac{1}{2}} e^{-\lambda x}\, L_n^{\pm p-\frac{1}{2}}(\lambda x)\, H_{2n}(\sqrt{\lambda x})\, dx = \frac{(-1)^n ((2n)!)^2 \sqrt{\pi}}{4^n (n!)^2\, (\frac{1}{2})_n\, \sqrt{\lambda}}, \quad (\lambda > 0, p:\ arbitrary). \tag{89}$$

Note also, if $c = 1/2$ and $p = 1$, it easily follows

$$\int_0^\infty x^{j-1} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n+1}(\sqrt{z\,x})\,dx$$
$$= (-1)^{m+n} \frac{(2m)!}{m!} \frac{2\sqrt{z}(2n+1)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^{\frac{1}{2}+j}} \sum_{k=0}^{m} \frac{(\frac{1}{2}+j)_k (-m)_k}{(\frac{1}{2})_k\, k!} \left(\frac{w}{\lambda}\right)^k$$
$$\times\, {}_2F_1(-n, \frac{1}{2}+j+k; \frac{3}{2}; \frac{z}{\lambda}),$$
$$(j > 0;\ \lambda > 0;\ m, n = 0, 1, \ldots), \tag{90}$$

For $j > n$ and $z = \lambda$

$$\int_0^\infty x^{j-1} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n+1}(\sqrt{\lambda\, x})\,dx = (-1)^{m+n} \frac{(2m)!}{m!} \frac{2(2n+1)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^j} \frac{(1-j)_n}{(\frac{3}{2})_n}$$
$$\times\, {}_3F_2(j+\frac{1}{2}, j, -m; \frac{1}{2}, j-n; \frac{w}{\lambda}) \tag{91}$$

and for $c = 1/2$ and $p = 0$

$$\int_0^\infty x^{j-\frac{1}{2}} e^{-\lambda x} H_{2m}(\sqrt{wx}) H_{2n}(\sqrt{z\,x})\,dx = (-1)^{m+n} \frac{(2m)!}{m!} \frac{(2n)!}{n!} \frac{\Gamma(\frac{1}{2}+j)}{\lambda^{\frac{1}{2}+j}}$$
$$\times \sum_{k=0}^{m} \frac{(\frac{1}{2}+j)_k (-m)_k}{(\frac{1}{2})_k\, k!} \left(\frac{w}{\lambda}\right)^k {}_2F_1(-n, \frac{1}{2}+j+k; \frac{1}{2}; \frac{z}{\lambda}) \tag{92}$$

We may remark that all the above results involving ${}_1F_1$ can be rewritten in the representation using the Whittaker function because of the following relationship:

$${}_1F_1(a, b, z) = e^{z/2} z^{-b/2} M_{(b-2a)/2,(b-1)/2}(z)$$

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References

[1] P. Appell. *Sur les fonctions hypergéometriques de plusieurs variables.* Gauthier-Villars, Paris, 1925. Mém. des Sciences Math. de l'Acad. des Sciences de Paris, III.

[2] Yu. A. Brychkov. *Handbook of special functions: derivatives, integrals, series and other formulas*, Chapman and Hall/CRC (2008).

[3] Von W. Gordon, *Zur Berechnung der Matrizen beim Wasserstoffatom*, Ann. Phys. Berlin **394** (1929) 1031.

[4] E. Karule, *Integrals of two confluent hypergeometric functions.* J. Phys. A: Math. Gen. **23** (1990) 1969.

[5] L. D. Landau and E. M. Lifshitz, *Quantum mechanics: Non-relativistic theory*, 3rd Edition, Pergamon Press (1965).

[6] S. B. Opps, N. Saad, H.M. Srivastava, *Recursion formulas for Appells hypergeometric function F_2 with some applications to radiation field problems*, Appl. Math. Comput. **207** (2009) 545.

[7] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev. *Integrals and series: Special functions*, Vol. 3. Gordon and Breach Science Pub., New York (1990).

[8] N. Saad and R. L. Hall, *Integrals containing confluent hypergeometric functions with applications to perturbed singular potentials*, J. Phys. A: Math. Gen. **36** (2003) 7771.

[9] L. J. Slater, Generalized Hypergeometric Functions (Cambridge: Cambridge University Press 1966).

[10] V. F. Tarasov, *W. Gordon's integral (1929) and its representations by means of Appell's functions F_2, F_1, and F_3,* J. Math. Phys. **44** (2003) 1449.

5 Appendix

In this appendix we summarize some recurrence relations of the Gordon's integral that follow using the contiguous relations of the confluent hypergeometric functions:

$$\mathrm{J}_{c+1}^{j(\pm p)}(b+1,b';\lambda,w,z) = \frac{c}{w}\left[\mathrm{J}_{c}^{j(\pm p)}(b+1,b';\lambda,w,z) - \mathrm{J}_{c}^{j(\pm p)}(b,b';\lambda,w,z)\right].$$

$$\begin{aligned}\mathrm{J}_{c}^{j(\pm p)}(b+1,b';\lambda,w,z) &= \frac{c-b}{b}\,\mathrm{J}_{c}^{j(\pm p)}(b-1,b';\lambda,w,z) + \frac{w}{b}\,\mathrm{J}_{c}^{(j+1)(\pm p)}(b,b';\lambda,w,z)\\ &+ \frac{2b-c}{b}\,\mathrm{J}_{c}^{j(\pm p)}(b,b';\lambda,w,z),\end{aligned}$$

$$\mathrm{J}_{c+1}^{j(\pm p)}(b+1,b';\lambda,w,z) = \frac{(c+1-k)_k}{w^k}\sum_{m=0}^{k}\frac{(-1)^m\,k!}{m!\,(k-m)!}\,\mathrm{J}_{c+1-k}^{j(\pm p+k)}(b+1-m,b';\lambda,w,z)$$

$$\mathrm{J}_{c}^{j(\pm p)}(b,b';\lambda,w,z) = \frac{b}{c}\,\mathrm{J}_{c+1}^{(j-1)(\pm p-1)}(b+1,b';\lambda,w,z) - \frac{b-c}{c}\,\mathrm{J}_{c+1}^{(j-1)(\pm p-1)}(b,b';\lambda,w,z),$$

$$\begin{aligned}\mathrm{J}_{c}^{j(\pm p)}(b+1,b';\lambda,w,z) &= \frac{b}{c}\,\mathrm{J}_{c+1}^{(j+1)(\pm p-1)}(b+1,b';\lambda,w,z) + \frac{w}{c}\,\mathrm{J}_{c+1}^{j(\pm p-1)}(b+1,b';\lambda,w,z)\\ &- \frac{b-c}{c}\,\mathrm{J}_{c+1}^{(j+1)(\pm p-1)}(b,b';\lambda,w,z),\end{aligned}$$

$$\begin{aligned}\mathrm{J}_{c}^{j(\pm p)}(b+1,b';\lambda,w,z) &= \mathrm{J}_{c}^{j(\pm p)}(b,b';\lambda,w,z) + \frac{w}{b}\,\mathrm{J}_{c}^{(j+1)(\pm p)}(b,b';\lambda,w,z)\\ &- \frac{w(b-c)}{cb}\,\mathrm{J}_{c+1}^{j(\pm p-1)}(b,b';\lambda,w,z),\end{aligned}$$

$$\begin{aligned}\mathrm{J}_{c-1}^{j(\pm p)}(b,b';\lambda,w,z) &= \mathrm{J}_{c}^{(j-1)(\pm p)}(b,b';\lambda,w,z) + \frac{w}{c-1}\,\mathrm{J}_{c}^{j(\pm p-1)}(b,b';\lambda,w,z)\\ &+ \frac{w(b-c)}{c(1-c)}\,\mathrm{J}_{c+1}^{(j-1)(\pm p-2)}(b,b';\lambda,w,z),\end{aligned}$$

$$\begin{aligned}\mathrm{J}_{c}^{j0}(b,b';\lambda,w,z) &= \frac{b1-c}{z}\,\mathrm{J}_{c}^{(j-1)0}(b,b'+1;\lambda,w,z) + \frac{b'-c}{z}\,\mathrm{J}_{c}^{(j-1)0}(b,b'-1;\lambda,w,z)\\ &+ \frac{c-2b'}{z}\,\mathrm{J}_{c}^{(j-1)0}(b,b';\lambda,w,z).\end{aligned}$$

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