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On The Solutions of Linear Matrix Quaternionic Equations and Their Systems

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Abstract

In this paper, our main aim is to investigate the solvability, existence of unique solution, closed-form solutions of some linear matrix quaternion equations with one unknown and of their systems with two unknowns. By means of the arithmetic operations on matrix quaternions, the linear matrix quaternion equations that is considered herein could be converted into four classical real linear equations, the solutions of the linear matrix quaternion equations are derived by solving four classical real linear equations based on the inverses and generalized inverses of matrices. Also, efficiency and accuracy of the presented method are shown by several examples.

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1 Introduction

The quaternion and quaternion matrices play a role in computer science, quantum physics, signal and color image processing, and so on (e.g., [1, 17, 22].

General properties of quaternion matrices can be found in [29].

Linear matrix equations are often encountered in many areas of computational mathematics, control and system theory. In the past several decades, solving matrix equations has been a hot topic in the linear algebraic field (see, for example, [2, 3, 12, 18, 19, 21 and 30]).

Several results on the solutions of some quaternionic and octonionic equations have been obtained. For example, the author of the paper [20] classified solutions of the quaternionic equation $ax+xb=c$. In [27], the linear equations of the forms $ax=xb$ and $ax=\overline{x}b$ in the real Cayley–Dickson algebras (quaternions, octonions, sedenions) are solved and form for the roots of such equations is established. In [13], the solutions of the equations of the forms $ax=xb$ and $ax=\overline{x}b$ for some generalizations of quaternions and octonions are investigated. In [23], the $\alpha x\beta+\gamma x\delta=\rho$ linear quaternionic equation with one unknown, $\alpha x\beta+\gamma x\delta=\rho$, is solved. In [6], Bolat and İpek first considered the linear octonionic equation with one unknown of the form $\alpha(x\alpha)=(\alpha x)\alpha=\alpha x\alpha=\rho$, with $0\neq\alpha\in\mathbf{O}$, second presented a method which is reduce this octonionic equation to an equation with the left and right coefficients to a real system of eight equations to find the solutions of this equation, and finally reached the solutions of this linear octonionic equation from this real system. In [7], Bolat and İpek obtained the solutions of some linear equations with two terms and one unknown by the method of matrix representations of complex quaternions over the complex quaternion field and to investigate the solutions of some complex quaternionic linear equations.

Equations considered in here have the origin in the classic papers by Tian [27] in 1999 and Shpakivskyi [23] in 2010 for studying some topics of quaternion and octonion field. Their pioneering works have lead to extensive research of quaternionic and octonionic equations, and many researchers have established relationships obtained by them to several mathematics subjects, such as differential equations and linear algebra (see, e.g., the papers [23-29] for more details about the quaternionic equation and the related topics).

Motivated by the work mentioned above, in this paper, we first define matrix quaternions and give some algebraic properties of matrix quaternions and a lemma together with its proof dealed with quaternions in this type. Then we unite and reflect upon some related issues that are crucial to the study of linear algebra over the quaternions: spectra, modules, and inner products. Finally we investigate the solvability, existence of unique solution, closed-form solutions of some linear matrix quaternion equations with one unknown and of their systems with two unknowns. One of the main techniques that is used herein is that of embeddings of the quaternions into real field. In this sense, then, algebra over quaternions is obtained from linear algebra over real numbers. We note that all our results are primarily of theoretical interest and we hope that they will lead to new insight into and better understanding of the relations between the most popular methods for solving matrix quaternionic and octonionic equations.

2 Preliminaries

In this section, we introduce some definitions, notations and basic properties which we need to use in the presentations and proofs of our main results.

We start by first recalling some basic results concerning Hamilton quaternion algebra $\mathbf{H}$, which can be found in classic books on this subject. For results concerning quaternion analysis we refer to [4, 9, 29].

The quaternion, which is a type of hypercomplex numbers, was formally introduced by Hamilton [10] in 1844. The definition of quaternion is

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k, \tag{1}$$

which obey the conventional algebraic rule for addition and multiplication by scalars (real numbers) and which obey an associative non-commutative rule for multiplication where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j, ijk = -1. \tag{2}$$

Throughout the paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbf{H} = \{q_0 + q_1 i + q_2 j + q_3 k : q_s \in \mathbb{R}, s = 0, 1, 2, 3\}, \tag{3}$$

where i, j and k satisfy the Eq.(2), by

$$Q_M^{m\times n} = \left\{A = A_0 + A_1 i + A_2 j + A_3 k : A_s \in \mathbb{R}^{m\times n}, s = 0, 1, 2, 3\right\}, \tag{4}$$

where i, j and k satisfy the Eq.(2). Next, several definitions and properties adopted in this paper are introduced to make the following sections readily comprehensible. More details can be found in [29].

It is frequently useful to regard quaternions as an ordered set of 4real quantities which we write as

$$\mathbf{q} = [q_0, q_1, q_2, q_3], \tag{5}$$

or as a combination of a scalar and a vector

$$\mathbf{q} = [q_0, q], \tag{6}$$

where $q = [q_1, q_2, q_3]$. A "scalar" quaternion has zero vector part and we shall write this as $[q_0, 0] = q_0 = 0$. A "pure" quaternion has zero scalar part $[0, q]$. In the scalar–vector representation, multiplication becomes

$$\mathbf{pq} = (p_0 q_0 - p \cdot q, p_0 q + q_0 p + p \times q), \tag{7}$$

where "·" and "×" are the vector dot and cross product. The conjugate of a quaternion is given by

$$\overline{\mathbf{q}}= [q_0, -q] \,, \tag{8}$$

the squared norm of a quaternion is

$$|\mathbf{q}|^2 = \mathbf{q}\overline{\mathbf{q}} = q_0^2 + q_1^2 + q_2^2 + q_3^2, \tag{9}$$

and its inverse is

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}. \tag{10}$$

Quaternions with $|\mathbf{q}| = 1$ are called as unit quaternions, for which we have $\mathbf{q}^{-1} = \overline{\mathbf{q}}$.

3 Main Results

In this section, we first define matrix quaternions and give some algebraic properties of matrix quaternions and a lemma together with its proof dealed with quaternions in this type. Then we unite and reflect upon some related issues that are crucial to the study of linear algebra over the quaternions: spectra, modules, and inner products. Finally we investigate the solvability, existence of unique solution, closed-form solutions of the some linear matrix quaternion equations with one unknown and of their systems with two unknowns.

Let $\mathbf{I}$ be the $n \times n$ identity matrix and let $\mathbf{H}$, $\mathbf{J}$, $\mathbf{K}$ be $n \times n$ matrices with real elements. Then the set of the matrix quaternions may be written in form as

$$M_Q^{m\times n} = \left\{ A = A_0\mathbf{I} + A_1\mathbf{H} + A_2\mathbf{J} + A_3\mathbf{K} : A_s \in \mathbb{R}^{m\times n}, s = 0, 1, 2, 3 \right\}, \tag{11}$$

and the rules of quaternion addition and multiplication will follow from those of matrix addition and multiplication provided that the matrices $\mathbf{H}$, $\mathbf{J}$, $\mathbf{K}$ satisfy the "Hamiltonian conditions" [8]:

$$\begin{aligned} \mathbf{HH}&= -\mathbf{I}, \quad \mathbf{JJ} = -\mathbf{I}, \quad \mathbf{KK} = -\mathbf{I}, \\ \mathbf{HJ}&= \mathbf{K}, \quad \mathbf{JK} = \mathbf{H}, \quad \mathbf{KH} = \mathbf{J}, \\ \mathbf{JH}&= -\mathbf{K}, \quad \mathbf{KJ} = -\mathbf{H}, \quad \mathbf{HK} = -\mathbf{J}. \end{aligned} \tag{12}$$

Addition and subctraction of the matrix quaternions $A = a_0\mathbf{I}+a_1\mathbf{H}+a_2\mathbf{J}+a_3\mathbf{K}$, $B = b_0\mathbf{I} + b_1\mathbf{H} + b_2\mathbf{J} + b_3\mathbf{K} \in M_Q^{n\times n}$ are

$$A \pm B = (a_0 \pm b_0)\mathbf{I} + (a_1 \pm b_1)\,\mathbf{H} + (a_2 \pm b_2)\,\mathbf{J} + (a_3 \pm b_3)\,\mathbf{K}. \tag{13}$$

Multiplication of the matrix quaternions A, $B \in M_Q^{n\times n}$ is given by

$$\begin{aligned} A.B &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)\,\mathbf{I} + (a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3)\,\mathbf{H} \\ &\quad + (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)\,\mathbf{J} \\ &\quad + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\,\mathbf{K} \end{aligned} \tag{14}$$

and hence in particular, multiplications of the matrix quaternions is not necessarily commutative. The conjugate of matrix quaternion $A \in M_Q^{n\times n}$ is

$$\overline{A} = a_0\mathbf{I} - a_1\mathbf{H} - a_2\mathbf{J} - a_3\mathbf{K} \tag{15}$$

and, therefore it is

$$A\ + \overline{A} = 2a_0\mathbf{I}, \overline{AB} = \overline{B}\overline{A}. \tag{16}$$

For $k \in \mathbb{R}$, the matrix $k.A$ is the matrix

$$k.A = ka_0\mathbf{I} + ka_1\mathbf{H} + ka_2\mathbf{J} + ka_3\mathbf{K} \in M_Q^{m\times n}. \tag{17}$$

Scalar product of the matrix quaternions $\alpha, \beta \in M_Q^{n\times n}$ is the scalar

$$\langle \alpha, \beta \rangle = (\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)\,. \tag{18}$$

The following Lemma has a key role in the remaining sections of this note.

Lemma 3.1 *For any $\alpha = \alpha_0\mathbf{I} + \alpha_1\mathbf{H} + \alpha_2\mathbf{J} + \alpha_3\mathbf{K}$, $\beta = \beta_0\mathbf{I} + \beta_1\mathbf{H} + \beta_2\mathbf{J} + \beta_3\mathbf{K} \in M_Q^{n\times n}$ the following equality is true:*

$$\alpha\beta = \beta\alpha - 2\overrightarrow{\beta}\,\overrightarrow{\alpha} - 2\,\langle \alpha, \beta \rangle\,\mathbf{I}. \tag{19}$$

Proof 3.2 *For α and β matrix quaternions in above Lemma 3.1, we obtain the following fundamental equalities for the expressions $\alpha\beta$, $\beta\alpha$ and $\overrightarrow{\beta}\,\overrightarrow{\alpha}$,*

$$\begin{aligned} \alpha\beta &= (\alpha_0\mathbf{I} + \alpha_1\mathbf{H} + \alpha_2\mathbf{J} + \alpha_3\mathbf{K})\,(\beta_0\mathbf{I} + \beta_1\mathbf{H} + \beta_2\mathbf{J} + \beta_3\mathbf{K}) \\ &= (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)\,\mathbf{I} \\ &\quad + (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)\,\mathbf{H} \\ &\quad + (\alpha_0\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_0 + \alpha_3\beta_1)\,\mathbf{J} \\ &\quad + (\alpha_0\beta_3 + \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_0)\,\mathbf{K}, \end{aligned} \tag{20}$$

$$\begin{aligned} \beta\alpha &= (\beta_0\mathbf{I} + \beta_1\mathbf{H} + \beta_2\mathbf{J} + \beta_3\mathbf{K})\,(\alpha_0\mathbf{I} + \alpha_1\mathbf{H} + \alpha_2\mathbf{J} + \alpha_3\mathbf{K}) \\ &= (\beta_0\alpha_0 - \beta_1\alpha_1 - \beta_2\alpha_2 - \beta_3\alpha_3)\,\mathbf{I} \\ &\quad + (\beta_1\alpha_0 + \beta_0\alpha_1 - \beta_3\alpha_2 + \beta_2\alpha_3)\,\mathbf{H} \\ &\quad + (\beta_2\alpha_0 + \beta_3\alpha_1 + \beta_0\alpha_2 - \beta_1\alpha_3)\,\mathbf{J} \\ &\quad + (\beta_3\alpha_0 - \beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3)\,\mathbf{K}, \end{aligned} \tag{21}$$

$$\begin{aligned}\overrightarrow{\beta}\overrightarrow{\alpha} &= (\beta_1\mathbf{H}+\beta_2\mathbf{J}+\beta_3\mathbf{K})(\alpha_1\mathbf{H}+\alpha_2\mathbf{J}+\alpha_3\mathbf{K}) \\ &= (-\beta_1\alpha_1-\beta_2\alpha_2-\beta_3\alpha_3)\mathbf{I} \\ &\quad +(\beta_2\alpha_3-\beta_3\alpha_2)\mathbf{H} \\ &\quad +(-\beta_1\alpha_3+\beta_3\alpha_1)\mathbf{J} \\ &\quad +(\beta_1\alpha_2-\beta_2\alpha_1)\mathbf{K},\end{aligned} \tag{22}$$

and

$$\langle\alpha,\beta\rangle\,\mathbf{I} = (\alpha_0\beta_0+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)\,\mathbf{I}. \tag{23}$$

Thus, from (20), (21), (22) *and* (23) *we reach the result of the Lemma.*

Lemma 3.1 is very intuitive and simple. By using this result, we are able to prove the following results regarding solutions of matrix quaternionic equations considered in Sections 3.1, 3.2 and 3.3.

3.1 The Linear Matrix Quaternionic Equations and Systems with one Addend

Applying Lemma 3.1 to a linear equation,

$$\alpha X\beta+\gamma X\delta=\rho, \tag{24}$$

where $\{\alpha,\beta,\gamma,\delta,\rho\}\subset M_Q^{n\times n}$, in matrix quaternions, we now reduce this equation to a real system with four unknown. For any $\alpha=\alpha_0\mathbf{I}+\alpha_1\mathbf{H}+\alpha_2\mathbf{J}+\alpha_3\mathbf{K}$, $\beta=\beta_0\mathbf{I}+\beta_1\mathbf{H}+\beta_2\mathbf{J}+\beta_3\mathbf{K}\in M_Q^{n\times n}$, using Lemma 3.1 we obtain

$$\begin{aligned}\alpha X\beta &= \alpha\left(\beta X-2\overrightarrow{\beta}\overrightarrow{X}-2\left\langle\overrightarrow{\beta},\overrightarrow{X}\right\rangle\mathbf{I}\right) \\ &= \alpha\beta X-2\alpha\overrightarrow{\beta}\overrightarrow{X}-2\alpha\left\langle\overrightarrow{\beta},\overrightarrow{X}\right\rangle\mathbf{I} \\ &= \alpha\beta\left(X_0\mathbf{I}+\overrightarrow{X}\right)-2\alpha\overrightarrow{\beta}\overrightarrow{X}-2\alpha\left\langle\overrightarrow{\beta},\overrightarrow{X}\right\rangle\mathbf{I} \\ &= \alpha\beta X_0+\left(\alpha\beta-2\alpha\overrightarrow{\beta}\right)\overrightarrow{X}-2\alpha\left\langle\overrightarrow{\beta},\overrightarrow{X}\right\rangle\mathbf{I}.\end{aligned}$$

Then, similarly, for $\gamma X\delta$ we have

$$\gamma X\delta=\gamma\delta X_0+\left(\gamma\delta-2\gamma\overrightarrow{\delta}\right)\overrightarrow{X}-2\gamma\left\langle\overrightarrow{\delta},\overrightarrow{X}\right\rangle\mathbf{I}. \tag{25}$$

Let $\rho=\rho_0\mathbf{I}+\rho_1\mathbf{H}+\rho_2\mathbf{J}+\rho_3\mathbf{K}$. Then, from the last two equalities, equation (24) is equivalent to the following:

$$\begin{aligned}\alpha X\beta+\gamma X\delta &= (\alpha\beta+\gamma\delta)X_0+\left[\left(\alpha\beta-2\alpha\overrightarrow{\beta}+\gamma\delta-2\gamma\overrightarrow{\delta}\right)\mathbf{H}-2\alpha\beta_1-2\gamma\delta_1\right]X_1 \\ &\quad +\left[\left(\alpha\beta-2\alpha\overrightarrow{\beta}+\gamma\delta-2\gamma\overrightarrow{\delta}\right)\mathbf{J}-2\alpha\beta_2-2\gamma\delta_2\right]X_2\end{aligned}$$

$$\begin{aligned}
& + \left[\left(\alpha\beta - 2\alpha\overrightarrow{\beta} + \gamma\delta - 2\gamma\overrightarrow{\delta}\right)\mathbf{K} - 2\alpha\beta_3 - 2\gamma\delta_3\right] X_3 \\
= \;& \chi X_0 + \varphi X_1 + \psi X_2 + \phi X_3 \\
= \;& [\chi_0\mathbf{I} + \chi_1\mathbf{H} + \chi_2\mathbf{J} + \chi_3\mathbf{K}] X_0 + [\varphi_0\mathbf{I} + \varphi_1\mathbf{H} + \varphi_2\mathbf{J} + \varphi_3\mathbf{K}] X_1 \\
& + [\psi_0\mathbf{I} + \psi_1\mathbf{H} + \psi_2\mathbf{J} + \psi_3\mathbf{K}] X_2 + [\phi_0\mathbf{I} + \phi_1\mathbf{H} + \phi_2\mathbf{J} + \phi_3\mathbf{K}] X_3 \\
= \;& \rho_0\mathbf{I} + \rho_1\mathbf{H} + \rho_2\mathbf{J} + \rho_3\mathbf{K}. \qquad (26)
\end{aligned}$$

Therefore, we obtain that the equation (24) is equivalent to the system

$$\begin{aligned}
\chi_0 X_0 + \varphi_0 X_1 + \psi_0 X_2 + \phi_0 X_3 &= \rho_0, \\
\chi_1 X_0 + \varphi_1 X_1 + \psi_1 X_2 + \phi_1 X_3 &= \rho_1, \\
\chi_2 X_0 + \varphi_2 X_1 + \psi_2 X_2 + \phi_2 X_3 &= \rho_2, \qquad (27)\\
\chi_3 X_0 + \varphi_3 X_1 + \psi_3 X_2 + \phi_3 X_3 &= \rho_3,
\end{aligned}$$

so that the problem has been reduced to solution of the 4×4 real linear system.

Example 3.3 *Consider the following equation:*

$$(\mathbf{H} - 2\mathbf{K}) X\mathbf{J} + (\mathbf{I} - \mathbf{J}) X (\mathbf{H} + \mathbf{K}) = \mathbf{I} + \mathbf{4H} + \mathbf{5J} + \mathbf{6K}, \qquad (28)$$

in $M_Q^{n\times n}$. *For* $\alpha = \mathbf{H} - 2\mathbf{K}$, $\beta = \mathbf{J}$, $\gamma = \mathbf{I} - \mathbf{J}$, $\delta = \mathbf{H} + \mathbf{K}$ *and* $\rho = \mathbf{I} + \mathbf{4H} + \mathbf{5J} + \mathbf{6K}$, *this equation is of the form* (24). *Here* $\alpha\beta = \alpha\overrightarrow{\beta} = (\mathbf{H} - 2\mathbf{K})\mathbf{J} = \mathbf{K} + \mathbf{2H}$, $\gamma\delta = \gamma\overrightarrow{\delta} = (\mathbf{I} - \mathbf{J})(\mathbf{H} + \mathbf{K}) = 2\mathbf{K}$, $\alpha\beta_1 = \alpha\beta_3 = \gamma\delta_2 = 0$, $\alpha\beta_2 = (\mathbf{H} - 2\mathbf{K})\mathbf{I} = \mathbf{H} - 2\mathbf{K}$, $\gamma\delta_1 = \gamma\delta_3 = (\mathbf{I} - \mathbf{J})\mathbf{I} = \mathbf{I} - \mathbf{J}$. *Then from equality* (26) *we have:*

$$(2\mathbf{H} + 3\mathbf{K}) X_0 + (-\mathbf{J}) X_1 + (\mathbf{H} + 2\mathbf{K}) X_2 + (\mathbf{I} + 4\mathbf{J}) X_3 = \mathbf{I} + \mathbf{4H} + \mathbf{5J} + \mathbf{6K}, (29)$$

that is equivalent to the system

$$\begin{aligned}
X_3 &= 1, \\
2X_0 + X_2 &= 4, \\
-X_1 + 4X_3 &= 5, \\
3X_0 + 2X_2 &= 6.
\end{aligned}$$

The last system has one solution: $X_0 = 2$, $X_1 = -1$, $X_2 = 0$, $X_3 = 1$, *then* $X = 2\mathbf{I} - \mathbf{H} + \mathbf{K}$.

3.2 The Linear Matrix Quaternionic Systems with Two Addends

In this section, we consider a system of linear matrix quaternionic equations of the form

$$\begin{aligned}
AYB + CXD &= E \qquad (30)\\
PYQ + RXS &= T
\end{aligned}$$

where $A\neq\mathbf{0}$, $B\neq\mathbf{0}$, C,D,E,P,Q,R,S,T are given matrix quaternions and X, Y are unknown.

Let denote $\|A\|=\sqrt{a_0^2+a_1^2+a_2^2+a_3^2}$ and $\|B\|=\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}$.If we multiply the first equation in (30) by conjugate $\overline{A}$ on the left and by $\overline{B}$ on the right, from $\overline{A}A=\|A\|^2\mathbf{I}$ and $\overline{B}B=\|B\|^2\mathbf{I}$ we have

$$\|A\|^2 Y\|B\|^2\mathbf{I}+\overline{A}CXD\overline{B}=\overline{A}E\overline{B} \tag{31}$$

or

$$\|A\|^2 Y\|B\|^2+\overline{A}CXD\overline{B}=\overline{A}E\overline{B}. \tag{32}$$

Since $A\neq\mathbf{0}$, $B\neq\mathbf{0}$, we write $\|A\|^2\neq\mathbf{0}$ and $\|B\|^2\neq\mathbf{0}$. If we now multiply the equation (32) by $\frac{1}{\|A\|^2\|B\|^2}$, we get

$$Y=\frac{1}{\|A\|^2\|B\|^2}\overline{A}E\overline{B}-\frac{1}{\|A\|^2\|B\|^2}\overline{A}CXD\overline{B}. \tag{33}$$

Putting (33) into the second equation of system (30) we get

$$\frac{1}{\|A\|^2\|B\|^2}P\overline{A}E\overline{B}Q-\frac{1}{\|A\|^2\|B\|^2}P\overline{A}CXD\overline{B}Q+RXS=T, \tag{34}$$

and hence the equation (34) is of the form (24) with

$$\alpha=-\frac{1}{\|A\|^2\|B\|^2}P\overline{A}C,\beta=D\overline{B}Q,\rho=T-\frac{1}{\|A\|^2\|B\|^2}P\overline{A}E\overline{B}Q. \tag{35}$$

From equation(34) and equation(35) , we find X. Then from equation (33), we calculate Y.

Example 3.4 *Consider the following system:*

$$\begin{cases}\mathbf{H}Y\mathbf{J}+\mathbf{J}X\mathbf{H}=\mathbf{K},\\ \mathbf{K}Y\mathbf{H}+\mathbf{H}X\mathbf{K}=\mathbf{I}+\mathbf{H}+\mathbf{J}\end{cases}, \tag{36}$$

in $M_Q^{n\times n}$. This is a system of the form (30) *. From* (26) *, we get the equation allowing to find X :*

$$-2X_2\mathbf{I}-\mathbf{2}X_3\mathbf{H}=\mathbf{I}+\mathbf{H}. \tag{37}$$

Then $X=X_0\mathbf{I}+X_1\mathbf{H}-\frac{1}{2}\mathbf{J}-\frac{1}{2}\mathbf{K}$, for any X_0,$X_1\in\mathbb{R}$. From formula (33) *, we obtain:*

$$\begin{aligned}Y &= \mathbf{HKJ}-\mathbf{HJ}\left(X_0\mathbf{I}+X_1\mathbf{H}-\frac{1}{2}\mathbf{J}-\frac{1}{2}\mathbf{K}\right)\mathbf{HJ}\\ &= (1+X_0)\mathbf{I}-X_1\mathbf{H}-\frac{1}{2}\mathbf{J}-\frac{1}{2}\mathbf{K}, for any X_0,X_1\in\mathbb{R}.\end{aligned}$$

In that case, the solution of the sytem (36) *has the form*

$$\begin{cases}X=X_0\mathbf{I}+X_1\mathbf{H}-\frac{1}{2}\mathbf{J}-\frac{1}{2}\mathbf{K},\\ Y=(1+X_0)\mathbf{I}-X_1\mathbf{H}-\frac{1}{2}\mathbf{J}-\frac{1}{2}\mathbf{K},\end{cases} \tag{38}$$

for any $X_0,X_1\in\mathbb{R}$.

3.3 The General Linear Matrix Quaternionic Equation with One Unknown and Systems of Equations with Two Unknowns

In this section ,we search for a solution of the general linear matrix quaternionic equation:

$$\sum_{s=1}^{N} \alpha_s X \beta_s = \rho. \tag{39}$$

We can write the equation to which (39) is equivalent:

$$AX_0+[(A-2B)\,\mathbf{H}-2C_1]\,X_1+[(A-2B)\,\mathbf{J}-2C_2]\,X_2+[(A-2B)\,\mathbf{K}-2C_3]\,X_3=\rho, \tag{40}$$

where $A=\sum\limits_{s=1}^{N}\alpha_s\beta_s$, $B=\sum\limits_{s=1}^{N}\alpha_s\overrightarrow{\beta_s}$, $C_q=\sum\limits_{s=1}^{N}\alpha_s\beta_s^{(q)}$, $q=1,2,3$. If all $\beta_s\in\overrightarrow{M_Q^{n\times n}}$, then $A=B$.

We denote

$$\begin{aligned} A &= A_0\mathbf{I}+A_1\mathbf{H}+A_2\mathbf{J}+A_3\mathbf{K}, \\ (A-2B)\,\mathbf{H}-2C_1 &= \lambda=\lambda_0\mathbf{I}+\lambda_1\mathbf{H}+\lambda_2\mathbf{J}+\lambda_3\mathbf{K}, \\ (A-2B)\,\mathbf{J}-2C_2 &= \mu=\mu_0\mathbf{I}+\mu_1\mathbf{H}+\mu_2\mathbf{J}+\mu_3\mathbf{K}, \\ (A-2B)\,\mathbf{K}-2C_3 &= \upsilon=\upsilon_0\mathbf{I}+\upsilon_1\mathbf{H}+\upsilon_2\mathbf{J}+\upsilon_3\mathbf{K}. \end{aligned} \tag{41}$$

Then equation (39) is equivalent to the system

$$\begin{cases} A_0X_0+\lambda_0X_1+\mu_0X_2+\upsilon_0X_3=\rho_0, \\ A_1X_0+\lambda_1X_1+\mu_1X_2+\upsilon_1X_3=\rho_1, \\ A_2X_0+\lambda_2X_1+\mu_2X_2+\upsilon_2X_3=\rho_2, \\ A_3X_0+\lambda_3X_1+\mu_3X_2+\upsilon_3X_3=\rho_3. \end{cases} \tag{42}$$

We now note the following proposition which is can directly be derived from (39).

Proposition 3.5 *If* $X=X_0+\overrightarrow{X}$ *is a root of equation* (39)*, then:*

1. $(-X)$ *is a root of equation* (39) *if and only if* $\rho=\mathbf{0}$;
2. *the conjugate* $\overline{X}$ *is a root of equation* (39) *if and only if* $AX_0=\rho$.

Proof 3.6 *The proof follows from* (40) .

Consider any system of linear quaternionic matrix equations of the form

$$\begin{cases} AYB+\sum\limits_{p=1}^{n}C_pXD_p=E, \\ \sum\limits_{m=1}^{r}F_mYG_m+\sum\limits_{t=1}^{l}\gamma_tX\delta_t=H, \end{cases} \tag{43}$$

where $A \neq \mathbf{0}$, $B \neq \mathbf{0}$, C_p, D_p, E, F_m, G_m, γ_t, δ_t, H be known matrix quaternions, X, Y are unknown.

From the first equation of the equation (43) we can find Y as an expression with X and substitute it into the second equation of the (43), then for X we have an equation of the form (39) with $N = rn + l$. From (42) we find X, and then we find Y. Let the following system of general form be given:

$$\begin{cases} \sum\limits_{\tau=1}^{v} A_\tau Y B_\tau + \sum\limits_{p=1}^{n} C_p X D_p = E, \\ \sum\limits_{m=1}^{r} F_m Y G_m + \sum\limits_{t=1}^{l} \gamma_t X \delta_t = H. \end{cases} \tag{44}$$

In order to solve system (44), we write every addend $\sum\limits_{\tau=1}^{v} A_\tau Y B_\tau$, $\sum\limits_{p=1}^{n} C_p X D_p$, $\sum\limits_{m=1}^{r} F_m Y G_m$, $\sum\limits_{t=1}^{l} \gamma_t X \delta_t$ in the form similar to (40) :

$$\begin{aligned} \sum_{\tau=1}^{v} A_\tau Y B_\tau = \widetilde{A} Y_0 &+ \left[\left(\widetilde{A} - 2\widetilde{B}\right)\mathbf{H} - 2\widetilde{C_1}\right] Y_1 + \left[\left(\widetilde{A} - 2\widetilde{B}\right)\mathbf{J} - 2\widetilde{C_2}\right] Y_2 \\ &+ \left[\left(\widetilde{A} - 2\widetilde{B}\right)\mathbf{K} - 2\widetilde{C_3}\right] Y_3, \end{aligned} \tag{45}$$

where $\widetilde{A} = \sum\limits_{\tau=1}^{v} A_\tau B_\tau$, $\widetilde{B} = \sum\limits_{\tau=1}^{v} A_\tau \overrightarrow{B_\tau}$, $\widetilde{C_q} = \sum\limits_{\tau=1}^{v} A_\tau B_\tau^{(q)}$, $q = 1, 2, 3$,

$$\begin{aligned} \sum_{p=1}^{n} C_p X D_p = \widehat{A} Y_0 &+ \left[\left(\widehat{A} - 2\widehat{B}\right)\mathbf{H} - 2\widehat{C_1}\right] Y_1 + \left[\left(\widehat{A} - 2\widehat{B}\right)\mathbf{J} - 2\widehat{C_2}\right] Y_2 \\ &+ \left[\left(\widehat{A} - 2\widehat{B}\right)\mathbf{K} - 2\widehat{C_3}\right] Y_3, \end{aligned} \tag{46}$$

where $\widehat{A} = \sum\limits_{p=1}^{n} C_p D_p$, $\widehat{B} = \sum\limits_{p=1}^{n} C_p \overrightarrow{D_p}$, $\widehat{C_q} = \sum\limits_{p=1}^{n} C_p D_p^{(q)}$, $q = 1, 2, 3$,

$$\begin{aligned} \sum_{m=1}^{r} F_m Y G_m = A' Y_0 &+ \left[(A' - 2B')\mathbf{H} - 2C_1'\right] Y_1 + \left[(A' - 2B')\mathbf{J} - 2C_2'\right] Y_2 \\ &+ \left[(A' - 2B')\mathbf{K} - 2C_3'\right] Y_3, \end{aligned} \tag{47}$$

where $A' = \sum\limits_{m=1}^{r} F_m G_m$, $B' = \sum\limits_{m=1}^{r} F_m \overrightarrow{G_m}$, $C_q' = \sum\limits_{m=1}^{r} F_m G_m^{(q)}$, $q = 1, 2, 3$,

$$\begin{aligned} \sum_{t=1}^{l} \gamma_t X \delta_t = A'' Y_0 &+ \left[(A'' - 2B'')\mathbf{H} - 2C_1''\right] Y_1 + \left[(A'' - 2B'')\mathbf{J} - 2C_2''\right] Y_2 \\ &+ \left[(A'' - 2B'')\mathbf{K} - 2C_3''\right] Y_3, \end{aligned} \tag{48}$$

where $A'' = \sum\limits_{t=1}^{l} \gamma_t \delta_t$, $B'' = \sum\limits_{t=1}^{l} \gamma_t \overrightarrow{\delta_t}$, $C''_q = \sum\limits_{t=1}^{l} \gamma_t \delta_t^{(q)}$, $q = 1, 2, 3$.

We shall introduce notations similar to those of (41) , then each of the equations of system (44) will be equivalent to a system of four equations with eight unknowns. Thus, the system (44) with two matrix quaternionic unknowns X, Y is equivalent to a system of eight equations with eight real unknowns $X_\eta, Y_\eta,\ \ \eta = 0, 1, 2, 3$.

Finally, we mention two applications of these results.

Example 3.7 *Consider the following system:*

$$\begin{cases} Y - \mathbf{H}X\mathbf{J} - \mathbf{J}X(\mathbf{H} + \mathbf{J}) = \mathbf{0}, \\ \mathbf{K}Y\mathbf{J} + (\mathbf{H} - \mathbf{K})Y\mathbf{H} + \mathbf{J}X\mathbf{K} + \mathbf{K}X\mathbf{J} = 23\mathbf{I} \end{cases}, \tag{49}$$

in $M_Q^{n\times n}$. *This is a system in the form* (43). *From the first equation of* (43) *we find* Y *and substitute it into the second equation of* (43). *Then we have the following equation:*

$$\begin{aligned} 23\mathbf{I} &= -\mathbf{J}X + \mathbf{H}X(\mathbf{I} - \mathbf{K}) \\ &+ (\mathbf{I} + \mathbf{J})X\mathbf{K} + (\mathbf{H} + \mathbf{K})X(-\mathbf{I} - \mathbf{K}) \\ &+ \mathbf{J}X\mathbf{K} + \mathbf{K}X\mathbf{J} \end{aligned} \tag{50}$$

We calculate coefficients of the equality (40) *for the case of* (49)*:*

$$\begin{aligned} A &= -\mathbf{J} + \mathbf{H}(\mathbf{I} - \mathbf{K}) + (\mathbf{I} + \mathbf{J})\mathbf{K} + (\mathbf{H} + \mathbf{K})(-\mathbf{I} - \mathbf{K}) + \mathbf{J}\mathbf{K} + \mathbf{K}\mathbf{J} = \mathbf{I} + \mathbf{H} + \mathbf{J}, \\ B &= -\mathbf{J} - \mathbf{H}\mathbf{K} + (\mathbf{I} + \mathbf{J})\mathbf{K} + (\mathbf{H} + \mathbf{K})\mathbf{K} + \mathbf{J}\mathbf{K} + \mathbf{K}\mathbf{J} = \mathbf{I} + \mathbf{H} + \mathbf{J} + \mathbf{K}, \\ C_1 &= C_2 = \mathbf{0}, C_3 = \mathbf{I} - \mathbf{2H} + \mathbf{J} - \mathbf{K}. \end{aligned}$$

Then to find X *we use the following system (obtained by* (41) , (42)*):*

$$\begin{cases} X_0 + X_1 + X_2 = 23, \\ X_0 - X_1 + 2X_2 + 3X_3 = 0, \\ X_0 - 2X_1 - X_2 - X_3 = 0, \\ X_1 - X_2 + X_3 = 0. \end{cases} \tag{51}$$

Consequently, it has one solution: $X_0 = 13$, $X_1 = 7$, $X_2 = 3$, $X_3 = -4$. *That is* $X = 13\mathbf{I} + 7\mathbf{H} + 3\mathbf{J} - 4\mathbf{K}$. *Then* $Y = \mathbf{H}X\mathbf{J} + \mathbf{J}X(\mathbf{H} + \mathbf{J}) = -13\mathbf{I} + \mathbf{H} - 17\mathbf{J} - 4\mathbf{K}$.

Example 3.8 *Consider the following system:*

$$\begin{cases} \mathbf{0} = \mathbf{H}Y\mathbf{J} + \mathbf{J}Y\mathbf{H} + \mathbf{K}X\mathbf{H} + \mathbf{H}X\mathbf{K}, \\ 2\mathbf{K} = (\mathbf{I} + \mathbf{H})Y(\mathbf{I} + \mathbf{J}) + (\mathbf{I} + \mathbf{J})Y(\mathbf{I} + \mathbf{H}) + (\mathbf{I} + \mathbf{K})X(\mathbf{I} + \mathbf{H}) \\ \qquad + (\mathbf{I} + \mathbf{H})X(\mathbf{I} + \mathbf{K}). \end{cases} \tag{52}$$

in $M_Q^{n\times n}$. *This is a system of the form* (44) .

For the first two addends of the first equation we calculate coefficients: $\widetilde{A}=\widetilde{B}=\mathbf{0},\ \widetilde{C_1}=\mathbf{J},\ \ \widetilde{C_2}=\mathbf{H},\ \ \widetilde{C_3}=\mathbf{0}.$

For the third and fourth addends of the first equation we calculate coefficients: $\widehat{A}=\widehat{B}=\mathbf{0},\ \widehat{C_1}=\mathbf{K},\ \ \widehat{C_2}=\mathbf{0},\ \ \widehat{C_3}=\mathbf{H}.$

For the first two addends of the second equation we calculate coefficients: $A'=2\mathbf{I}+2\mathbf{H}+2\mathbf{J},\ B'=\mathbf{H}+\mathbf{J},\ C_1'=\mathbf{I}+\mathbf{J},\ \ C_2'=\mathbf{I}+\mathbf{H},\ \ C_3'=\mathbf{0}.$

For the third and fourth addends of the second equation we calculate coefficients: $A''=2\mathbf{I}+2\mathbf{H}+2\mathbf{K},\ B''=\mathbf{H}+\mathbf{K},\ C_1''=\mathbf{I}+\mathbf{K},\ \ C_2''=\mathbf{0},\ \ C_3'=\mathbf{I}+\mathbf{H}.$

Then similarly to (40)*, we have a system with two equations:*

$$\left\{\begin{array}{c} \mathbf{0}=-2\mathbf{J}Y_1-2\mathbf{H}Y_2-2\mathbf{K}X_1-2\mathbf{H}X_3,\\ 2\mathbf{K}=\left(2\mathbf{I}+2\mathbf{H}+2\mathbf{J}\right)Y_0+\left(-2\mathbf{I}+2\mathbf{H}-2\mathbf{J}\right)Y_1\\ +\left(-2\mathbf{I}-2\mathbf{H}+2\mathbf{J}\right)Y_2+2\mathbf{K}Y_3\\ +\left(2\mathbf{I}+2\mathbf{H}+2\mathbf{K}\right)X_0\\ +\left(-2\mathbf{I}+2\mathbf{H}-2\mathbf{K}\right)X_1\\ +2\mathbf{J}X_2+\left(-2\mathbf{I}-2\mathbf{H}+2\mathbf{K}\right)X_3. \end{array}\right. \tag{53}$$

From the last system we reach the following system with eight real unknowns:

$$\left\{\begin{array}{c} Y_2+X_3=0,\\ Y_1=X_1=0,\\ Y_0-Y_1-Y_2+X_0-X_1-X_3=0,\\ Y_0+Y_1-Y_2+X_0+X_1-X_3=0,\\ Y_0-Y_1+Y_2+X_2=0,\\ Y_3+X_0-X_1+X_3=1. \end{array}\right. \tag{54}$$

Therefore, the one solution of the system (52) *and accordingly of the system* (52) *is*

$$\begin{aligned} X &= \left(1-X_3-Y_3\right)\mathbf{I}+\left(1-Y_3\right)\mathbf{J}+X_3\mathbf{K},\\ Y &= \left(X_3+Y_3-1\right)\mathbf{I}-X_3\mathbf{J}+Y_3\mathbf{K} \end{aligned}$$

for any $X_3, Y_3\in\mathbb{R}$.

4 Conclusions

In this paper, a new computational method based on the embeddings of the quaternions into real field was proposed for solving some linear matrix quaternion equations. Such problems can be transformed into linear real systems of algebraic equations which can be directly solved by the inverses and generalized inverses of matrices. Applicability and accuracy of the proposed method were checked on some examples.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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