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On the sequence related to Lucas numbers and its properties

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Abstract

The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this article, we study a new generalization $\{L_{k,n}\}$, with initial conditions $L_{k,0} = 2$ and $L_{k,1} = 1$, which is generated by the recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ for $n \geq 2$, where k is integer number. Some well-known sequence are special case of this generalization. The Lucas sequence is a special case of $\{L_{k,n}\}$ with $k = 1$. Modified Pell-Lucas sequence is $\{L_{k,n}\}$ with $k = 2$. We produce an extended Binet's formula for $\{L_{k,n}\}$ and, thereby, identities such as Cassini's, Catalan's, d'Ocagne's, etc. using matrix algebra. Moreover, we present sum formulas concerning this new generalization.

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1 Introduction

In recent years, many interesting properties of classic Fibonacci numbers, classic Lucas numbers and their generalizations have been shown by researchers and applied to almost every field of science and art. For the rich and related applications of these numbers, one can refer to the nature and different areas of the science [3-11]. The classic Fibonacci $\{F_n\}_{n\in\mathbb{N}}$ and Lucas $\{L_n\}_{n\in\mathbb{N}}$ sequences are defined as, respectively,

$$F_0 = 0,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geq 2$$

and

$$L_0 = 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \ \text{ for } n \geq 2$$

where F_n and L_n, respectively, denotes the nth classic Fibonacci and Lucas numbers. Besides of the usual Fibonacci and Lucas numbers, many kinds of generalizations of these numbers have been presented in the literature [3, 8, 9, 11]. In [3], the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, has been found by studying the recursive applications of two geometrical transformations used in the well-known four-triangle longest-edge(4TLE) partition and is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1},\ 1 \leq n, k \in \mathbb{Z}$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1.$$

In that paper, many properties of these numbers have been obtained directly from elementary matrix algebra. Many properties of these numbers have been deduced and related with the so-called Pascal 2-triangle [4]. Additionally the authors of [5] defined k-Fibonacci hyperbolic functions as similar to hyperbolic functions and Fibonacci hyperbolic functions. In [6], authors studied 3-dimensional k-Fibonacci spirals from a geometric point of view. m-extension of the Fibonacci and Lucas $p-$ numbers are defined in [8]. Afterwards, the continuous functions for the m-extension of the Fibonacci and Lucas p-numbers using the generalized Binet formulas have been obtained in that paper. The generating matrix, the Binet like formulas, applications to the coding theory and the generalized Cassini formula, i.e., of the Fibonacci $p-$numbers are given by Stakhov [10]. Stakhov and Rozin [9] showed that the formulas are similar to the Binet formulas given for the classical Fibonacci numbers, also defined to be of generalized Fibonacci and Lucas numbers or Fibonacci and Lucas p-numbers. As a similar study, in [10], it has been introduced the new continuous functions for the Fibonacci and Lucas p-numbers using Binet formulas. In [11], Civciv and Türkmen defined a new matrix generalization of the Fibonacci and Lucas numbers using essentially a matrix approach.

In this work, we define a new generalization of the classic Lucas sequence and give identities and sum formulas concerning this new generalization.

2 Main Results

Now, a new generalization of the classical Lucas sequence that its recurrence formula is depended on one parameter is introduced and some particular cases of this sequence are given.

Definition 1 *For any integer number $k \geq 1$, the kth Lucas sequence, say $\{L_{k,n}\}_{n\in N}$, is defined by*

$$L_{k,0} = 2,\ L_{k,1} = 1 \text{ and } L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } n \geq 2. \tag{1}$$

The following table summarizes some special cases of nth k-Lucas numbers $L_{k,n}$:

k	$L_{k,n}$
1	Lucas numbers
2	Modified Pell-Lucas numbers

In [3], many properties of $k-$ Fibonacci numbers from $M = \begin{bmatrix} k-1 & 1 \\ k & 1 \end{bmatrix}$ using matrix algebra are obtained. Matrix methods are very useful tools to solve many problems for stemming from number theory. Now we will obtain some algebraic properties of $k-$Lucas numbers via the matrix M.

The follow proposition gives that the elements of the first row of nth power of M are k-Fibonacci and Lucas numbers.

Proposition 1 *Let $M = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$. For any integer $n \geq 1$ holds:*

$$M^n = \begin{pmatrix} L_{k,n+1} - 3F_{k,n} & L_{k,n} - 2F_{k,n-1} \\ * & * \end{pmatrix}.$$

Proof. By induction: for $n = 1$:

$$M = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix} = \begin{pmatrix} L_{k,2} - 3F_{k,1} & L_{k,1} - 2F_{k,0} \\ * & * \end{pmatrix}$$

since $F_{k,0} = 0,\ F_{k,1} = 1, L_{k,1} = 1$ and $L_{k,2} = k + 2$. Let us suppose that the formula is true for $n - 1$:

$$M^{n-1} = \begin{pmatrix} L_{k,n} - 3F_{k,n-1} & L_{k,n-1} - 2F_{k,n-2} \\ * & * \end{pmatrix}.$$

Then,

$$\begin{aligned} M^n &= M^{n-1}M \\ &= \begin{pmatrix} L_{k,n} - 3F_{k,n-1} & L_{k,n-1} - 2F_{k,n-2} \\ * & * \end{pmatrix} \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix} \\ &= \begin{pmatrix} L_{k,n+1} - 3F_{k,n} & L_{k,n} - 2F_{k,n-1} \\ * & * \end{pmatrix}. \end{aligned}$$

Thus, the proof is completed. ■

It is well known that Binet's formulas are very used in the Fibonacci numbers theory. We will give Binet's formula for k-Lucas numbers by a function of the roots $r_1$and r_2 of the charactersictic equation associated to the recurrence relation the Eq.(1):

$$r^2 = kr + 1 \tag{2}$$

Proposition 2 ***(Binet's Formula for*** k***- Lucas numbers).*** *The nth* k*-Lucas number is given by*

$$L_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2}, \quad r_1 > r_2 \tag{3}$$

where $X = \frac{2+r_1}{r_1}$ *and* $Y = \frac{2+r_2}{r_2}$.

Proof. [Proof 1] Let $M = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$. We get spectral decomposition of the M matrix. The characteristic polynomial of the M matrix is

$$\det(M - \lambda I) = \begin{vmatrix} k-1-\lambda & 1 \\ k & 1-\lambda \end{vmatrix}$$

which yields the two eigenvalues

$$\lambda_1 = \frac{k + \sqrt{k^2+4}}{2}, \ \lambda_2 = \frac{k - \sqrt{k^2+4}}{2}$$

of the matrix M. Hence, we write $r_1 = \lambda_1$, $r_2 = \lambda_2$. Therefore,

$$u_1 = (\frac{\lambda_1 - 1}{k}, 1)^T$$

is an eigenvector of the matrix M corresponding the eigenvalue λ_1. Similar computation shows that the other eigenvector corresponding the eigenvalue λ_2 is

$$u_2 = (\frac{\lambda_2 - 1}{k}, 1)^T.$$

Let $P = (u_1|u_2)$ denote the matrix whose columns are the vectors u_1, u_2. That is,

$$P = \begin{pmatrix} \frac{\lambda_1 - 1}{k} & \frac{\lambda_2 - 1}{k} \\ 1 & 1 \end{pmatrix}.$$

The inverse matrix of P is given by

$$P^{-1} = \begin{pmatrix} \frac{k}{\lambda_1 - \lambda_2} & \frac{2 - k + \lambda_1 - \lambda_2}{2(\lambda_1 - \lambda_2)} \\ \frac{-k}{\lambda_1 - \lambda_2} & \frac{k - 2 + \lambda_1 - \lambda_2}{2(\lambda_1 - \lambda_2)} \end{pmatrix}.$$

It follows that

$$P^{-1}MP = \Lambda,$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Thus, $M = P\Lambda P^{-1}$ and we can compute the integer powers of M easily:

$$\begin{aligned} M^n &= \left(P\Lambda P^{-1}\right)^n \\ &= \left(P\Lambda P^{-1}\right)\left(P\Lambda P^{-1}\right)\left(P\Lambda P^{-1}\right)...\left(P\Lambda P^{-1}\right) \\ &= P\Lambda(P^{-1}P)\Lambda(P^{-1}P)...(P^{-1}P)\Lambda P^{-1} \\ &= P\Lambda^n P^{-1}, \text{ by } P^{-1}P = I. \end{aligned}$$

Since Λ is a diagonal matrix, the integer powers of Λ are easily computed by the formula:

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}.$$

Therefore, the equation

$$M^n = P\Lambda^n P^{-1}$$

becomes

$$M^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}. \tag{4}$$

Now, multiplying both sides of the Eq.(4) by $v = (1, 0)^T$, we obtain the matrix equality

$$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5}$$

From Proposition1 and the Eq.(5) ,

$$\begin{pmatrix} L_{k,n+1} - 3F_{k,n} \\ * \end{pmatrix} = \begin{pmatrix} \frac{(\lambda_1 - 1)\lambda_1^n - (\lambda_2 - 1)\lambda_2^n}{\lambda_1 - \lambda_2} \\ * \end{pmatrix} \tag{6}$$

Thus, for $X = \frac{2+\lambda_1}{\lambda_1}$ and $Y = \frac{2+\lambda_2}{\lambda_2}$, we reach using equality of matrices from the Eq.(6)

$$L_{k,n+1} = \frac{X\lambda_1^n - Y\lambda_2^n}{\lambda_1 - \lambda_2}.$$

Thus, the proof is completed. ■

Proof. [Proof 2] The roots of the characteristic equation the Eq.(2) are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$, and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$. Note that, since $0 < k$, then

$$\begin{aligned} r_2 &< 0 < r_1 \text{ and } |r_2| < r_1, \\ r_1 + r_2 &= k, \; r_1 r_2 = -1, \text{ and } r_1 - r_2 = \sqrt{k^2+4}. \end{aligned} \tag{7}$$

Therefore, the general term of the k-Lucas sequence may be expressed in the form

$$L_{k,n} = C_1 r_1^n + C_2 r_2^n,$$

for some coefficients C_1 and C_2. The constant C_1 and C_2 are determined by the initial conditions

$$\begin{aligned} 2 &= C_1 + C_2 \\ 1 &= C_1 r_1 + C_2 r_2. \end{aligned}$$

Solving above equation system for C_1 and C_2 , we get $C_1 = \frac{1-2r_2}{r_1-r_2}$, $C_2 = \frac{1-2r_1}{r_1-r_2}$. Therefore, we write

$$L_{k,n} = \frac{1-2r_2}{r_1-r_2} r_1^n + \frac{1-2r_1}{r_1-r_2} r_2^n. \tag{8}$$

For $X = \frac{2+r_1}{r_1}$ and $Y = \frac{2+r_2}{r_2}$ obtained in the Eq.(8), we get

$$L_{k,n} = \frac{X r_1^n - Y r_2^n}{r_1 - r_2},$$

which completes the proof. ■

Proposition 3 *Let* $L_{k,0} = 2,\; L_{k,1} = 1$ *and* $A = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$. *Then*

$$\begin{bmatrix} L_{k,n+1} \\ L_{k,n} \end{bmatrix} = A^n \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix}.$$

Proof. Now, we will prove the proposition by mathematical induction. For $n = 1$:

$$\begin{aligned} \begin{bmatrix} L_{k,2} \\ L_{k,1} \end{bmatrix} &= \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix} \\ &= A \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix} \end{aligned}$$

since $L_{k,1} = 1$, $L_{k,1} = 1$ and $L_{k,2} = k + 2$. Let us suppose that the formula is true for $n - 1$:

$$\begin{bmatrix} L_{k,n} \\ L_{k,n-1} \end{bmatrix} = A^{n-1} \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix}.$$

Then,

$$\begin{aligned} A^n \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix} &= A.A^{n-1}. \begin{bmatrix} L_{k,1} \\ L_{k,0} \end{bmatrix} \\ &= \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L_{k,n} \\ L_{k,n-1} \end{bmatrix} \\ &= \begin{bmatrix} L_{k,n+1} \\ L_{k,n} \end{bmatrix}. \end{aligned}$$

■

Proposition 4

$$\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = r_1. \tag{9}$$

Proof. [Proof 1] By using the Eq.(3),

$$\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = \lim_{n\to\infty} \frac{\frac{Xr_1^n - Yr_2^n}{r_1 - r_2}}{\frac{Xr_1^{n-1} - Yr_2^{n-1}}{r_1 - r_2}} = \lim_{n\to\infty} \frac{r_1^n \left(X - Y \left(\frac{r_2}{r_1} \right)^n \right)}{r_1^{n-1} \left(X - Y \left(\frac{r_2}{r_1} \right)^n \frac{r_1}{r_2} \right)},$$

and taking into account that $\lim_{n\to\infty} \left(\frac{r_2}{r_1} \right)^n = 0$ since $|r_2| < r_1$, Proposition 5 is proved. ■

Proof. [Proof 2] $\{x_n\}_{n=1}^{\infty} = \left\{ \frac{L_{k,n}}{L_{k,n-1}} \right\}_{n=1}^{\infty}$ is convergent. Let this sequence converges to x real number. Since $\frac{L_{k,n+1}}{L_{k,n}} = k + \frac{L_{k,n-1}}{L_{k,n}}$ and $k > 0$, $x > 0$. Then, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} &= k + \lim_{n\to\infty} \frac{L_{k,n-1}}{L_{k,n}} \\ &= k + \frac{1}{\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}}}. \end{aligned} \tag{10}$$

This gives us the equation

$$x^2 - kx - 1 = 0.$$

for x. This equation has the single positive root

$$\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = r_1.$$

■

Proposition 5 *For $n \geq 0$ holds:*

$$\sum_{i=0}^{n} \binom{n}{i} k^i L_{ki} = L_{k,2n}.$$

Proof. Using the Eq.(3) , we get

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} k^i L_{ki} &= \sum_{i=0}^{n} \binom{n}{i} k^i \frac{Xr_1^i - Yr_2^i}{r_1 - r_2} \\
&= \frac{1}{r_1 - r_2} \left[X \sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - Y \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i \right] \\
&= \frac{1}{r_1 - r_2} \left[X (1 + kr_1)^n - Y (1 + kr_2)^n \right] \\
&= L_{k,2n}, \text{ by } r_1^2 = kr_1 + 1 \text{ and } r_2^2 = kr_2 + 1.
\end{aligned}$$

■

Proposition 6 *The equality*

$$L_{k,n-1} L_{k,n+1} - L_{k,n}^2 = (-1)^{n+1} (2k + 3)$$

holds.

Proof. Now, let us consider the 2×2 linear system:

$$\begin{aligned}
L_{k,n} x + L_{k,n-1} y &= L_{k,n+1} \\
L_{k,n+1} x + L_{k,n} y &= L_{k,n+2}.
\end{aligned}$$

Since $L_{k,n}^2 - L_{k,n-1} L_{k,n+1} \neq 0$, this system has a unique solution such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Therefore, by Cramer's rule, we have

$$y = \frac{\begin{vmatrix} L_{k,n} & L_{k,n+1} \\ L_{k,n+1} & L_{k,n+2} \end{vmatrix}}{\begin{vmatrix} L_{k,n} & L_{k,n-1} \\ L_{k,n+1} & L_{k,n} \end{vmatrix}} = 1.$$

Thus, $L_{k,n+2} L_{k,n} - L_{k,n+1}^2 = L_{k,n}^2 - L_{k,n-1} L_{k,n+1}$. That is, $L_{k,n+2} L_{k,n} - L_{k,n+1}^2 = -\left(L_{k,n-1} L_{k,n+1} - L_{k,n}^2 \right)$. Now, let $P_{k,n} = L_{k,n-1} L_{k,n+1} - L_{k,n}^2$. Then this relation give us:

$$P_{k,n-1} = -P_{k,n}, n \geq 1, \;\; P_{k,1} = L_{k,0} L_{k,2} - L_{k,1}^2 = 2k + 3.$$

Solving the recurrence relation, we get $P_{k,n} = (-1)^{n+1} (2k + 3)$. Thus, $L_{k,n-1} L_{k,n+1} - L_{k,n}^2 = (-1)^{n+1} (2k + 3)$, where $n \geq 1$. ■

A more general case can be given by the following proposition.

Proposition 7 *The equality*

$$L_{k,n-r}L_{k,n+r}-L_{k,n}^{2}=(-1)^{n-r}(2k+3)F_{k,r}^{2}$$

holds.

Proof. By using the Eq.(3), we get

$$\begin{aligned}
L_{k,n-r}L_{k,n+r}-L_{k,n}^{2} &= \left(\frac{Xr_1^{n-r}-Yr_2^{n-r}}{r_1-r_2}\right)\left(\frac{Xr_1^{n+r}-Yr_2^{n+r}}{r_1-r_2}\right)-\left(\frac{Xr_1-Yr_2}{r_1-r_2}\right)^{2}\\
&= \frac{XY}{(r_1-r_2)^{2}}\left[-r_1^{n-r}r_2^{n+r}-r_2^{n-r}r_1^{n+r}+2r_1^{n}r_2^{n}\right]\\
&= -\frac{(3+2k)}{(r_1-r_2)^{2}}\left[-(r_1r_2)^{n}\left(\frac{r_2}{r_1}\right)^{r}-(r_1r_2)^{n}\left(\frac{r_1}{r_2}\right)^{r}+2(r_1r_2)^{n}\right]\\
&= (3+2k)(-1)^{n-r}F_{k,r}^{2}.
\end{aligned}$$

■

For $r=1$ and $k=1$, we reach the identity

$$L_{n-1}L_{n+1}-L_{n}^{2}=5(-1)^{n+1},$$

which is a special statement of the Proposition 7.

Proposition 8 (d'Ocagne identity) *For $m>n$, we write*

$$L_{k,m}L_{k,n+1}-L_{k,m+1}L_{k,n}=XY(-1)^{n}F_{k,m-n}.$$

Proof. By using the Eq.(3), we get

$$\begin{aligned}
L_{k,m}L_{k,n+1}-L_{k,m+1}L_{k,n} &= \frac{Xr_1^{m}-Yr_2^{m}}{r_1-r_2}\frac{Xr_1^{n+1}-Yr_2^{n+1}}{r_1-r_2}-\frac{Xr_1^{m+1}-Yr_2^{m+1}}{r_1-r_2}\frac{Xr_1^{n}-Yr_2^{n}}{r_1-r_2}\\
&= \frac{XY\left(r_1^{m}r_2^{n}\right)(r_1-r_2)-XY\left(r_1^{n}r_2^{m}\right)(r_1-r_2)}{(r_1-r_2)^{2}}\\
&= XY(-1)^{n}F_{k,m-n}.
\end{aligned}$$

■

Proposition 9 *The following equality*

$$\sum_{i=0}^{n}L_{k,i}=\frac{1}{k}\left(L_{k,n+1}+L_{k,n}+2k-3\right)$$

holds.

Proof. Let $S_{k,n}=\sum_{i=0}^{n}L_{k,i}$. Then, for $X=\frac{2+r_1}{r_1}$ and $Y=\frac{2+r_2}{r_2}$, we get

$$\begin{aligned} S_{k,n} &= \frac{1}{r_1-r_2}\sum_{i=0}^{n}\left(Xr_1^i-Yr_2^i\right) \\ &= \frac{1}{r_1-r_2}\left[X\sum_{i=0}^{n}r_1^i-Y\sum_{i=0}^{n}r_2^i\right] \\ &= \frac{1}{k}\left(L_{k,n+1}+L_{k,n}+2F_{k,2}-3\right) \\ &= \frac{1}{k}\left(L_{k,n+1}+L_{k,n}+2k-3\right),\ \text{by } F_{k,2}=k. \end{aligned}$$

■

Proposition 10 *For each real number* p $(p>r_1)$, *the equation*

$$\sum_{j=1}^{\infty}\frac{L_{k,j}}{p^j}=\frac{p+2}{p^2-kp-1}$$

is satisfied.

Proof. Substituting the definiton of j th k-Lucas number in the left hand side of the equation gives

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{L_{k,j}}{p^j} &= \lim_{n\to\infty}\sum_{j=1}^{n}\frac{\frac{Xr_1^j-Yr_2^j}{r_1-r_2}}{p^j} \\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\frac{X\left(\frac{r_1}{p}\right)^j-Y\left(\frac{r_2}{p}\right)^j}{r_1-r_2} \\ &= \frac{p+2}{p^2-pk-1}. \end{aligned}$$

■

Proposition 11 *Let* $L_{k,n}$ *is the nth* $k-$*Lucas number. The following equalities are hold:*

i) $\sum_{i=0}^{n}L_{k,2i}=\frac{1}{k}\left(L_{k,2n+1}+2k-1\right)$

ii) $\sum_{i=0}^{n}L_{k,2i+1}=\frac{1}{k}\left(L_{k,2n+2}-2\right)$

iii) $\sum_{i=0}^{n}L_{k,4i+1}=\frac{1}{k^2(k^2+4)}\left(L_{k,4n+5}-L_{k,4n+1}-2k^3+k^2-4k\right).$

Proof. i) Using the Eq.(3) , we get

$$\begin{aligned}\sum_{i=0}^{n} L_{k,2i} &= \sum_{i=0}^{n} \frac{Xr_1^{2i} - Yr_2^{2i}}{r_1 - r_2} \\ &= \frac{1}{r_1 - r_2}\left[\sum_{i=0}^{n} Xr_1^{2i} - \sum_{i=0}^{n} Yr_2^{2i}\right] \\ &= \frac{1}{k}\left(L_{k,2n+1} + 2k - 1\right).\end{aligned}$$

The proof of (ii) and (iii) can be shown in a similar fashion. ■

Let $k = 1$ in i), ii) and iii), then we obtain respectively following equalities, which we known from Lucas sequences,

$$\begin{aligned}\sum_{i=0}^{n} L_{2i} &= L_{2n+1} + 1, \\ \sum_{i=0}^{n} L_{2i+1} &= L_{2n+2} - 2, \\ \sum_{i=0}^{n} L_{4i+1} &= \frac{1}{5}\left(L_{4n+5} - L_{4n+1} - 5\right).\end{aligned}$$

Proposition 12 *For arbitrary integers $m, n \geq 1$, we have*

$$\sum_{i=1}^{n} L_{k,mi} = \frac{L_{k,mn+m} - (-1)^m L_{k,mn} - L_{k,m} + 2(-1)^m}{r_1^m + r_2^m - 1 - (-1)^m}.$$

Proof. By using of Binet's Formula given in the Eq.(3) and taking into account that $r_1 - r_2 = k$ and $r_1 r_2 = -1$, it is obtained

$$\begin{aligned}\sum_{i=1}^{n} L_{k,mi} &= \sum_{i=1}^{n} \frac{Xr_1^{mi} - Yr_2^{mi}}{r_1 - r_2} \\ &= \frac{1}{r_1 - r_2}\left[X\sum_{i=1}^{n} r_1^{mi} - Y\sum_{i=1}^{n} r_2^{mi}\right] \\ &= \frac{L_{k,mn+m} - (-1)^m L_{k,mn} - L_{k,m} + 2(-1)^m}{r_1^m + r_2^m - 1 - (-1)^m}.\end{aligned}$$

■

For $k = 1$, we get

$$\sum_{i=1}^{n} L_{mi} = \frac{L_{mn+m} - (-1)^m L_{mn} - L_m + 2(-1)^m}{r_1^m + r_2^m - 1 - (-1)^m}.$$

The follow proposition give us the generating funtions for the k-Lucas sequences.

Proposition 13 *Let m and n be integers and $L_{k,n}$ is the nth k-Lucas number. Then the following equalities are valid:*

i) $\sum_{n=0}^{\infty} L_{k,n}x^n = \frac{2-2kx+x}{1-kx-x^2}$

ii) $\sum_{n=0}^{\infty} L_{k,n+1}x^n = \frac{1+2x}{1-kx-x^2}$

iii) $\sum_{n=0}^{\infty} L_{k,2n+2}x^n = \frac{L_{k,2}-2x}{1-(k^2+2)x+x^2}$

iv) $\sum_{n=0}^{\infty} L_{k,m+n}x^n = \frac{L_{k,m}+L_{k,m-1}x}{1-kx-x^2}$

Proof. i) Using the Eq.(3) , we get

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{k,n}x^n &= \sum_{n=0}^{\infty} \frac{Xr_1^n - Yr_2^n}{r_1 - r_2} x^n \\
&= \frac{1}{r_1 - r_2}\left[X\sum_{n=0}^{\infty} r_1^n x^n - Y\sum_{n=0}^{\infty} r_2^n x^n\right] \\
&= \frac{2-2kx+x}{1-kx-x^2}
\end{aligned}$$

which completes the proof of (i).

Similarly, (ii), (iii) and (iv) can be proven.

For $k = 1$, we obtain a special sum such that

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

■

3 Conclusions

The conclusions arising from the work are as follows:

1. Some new identities have been obtained for k-Lucas numbers.

2. The some identities and sum formulas for the k-Lucas numbers have been presented.

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