On the Riesz Space Decomposition Theorem

Junfeng Liu

Department of Mathematics
Guangdong Polytechnic Normal University
Guangzhou 510665, People′s Republic of China
junfengliu6688@163.com

Abstract

In this paper, we extend the famous Jordan Space Decomposition Theorem and Riesz Space Decomposition Theorem, and the proof of the result of this paper does not depend on two theorems above. Since Jordan′s Theorem and Riesz′s Theorem are two famous results in operator theory, the two theorems are often detailed in the same book. Using the result of this paper, one may modify the discussion of the Riesz′s Theorem, and omit the discussion of Jordan′s Theorem, then two theorems above are obtained as corollaries.

Mathematics Subject Classification: 47A15

Keywords: Invariant subspace, bounded linear operator, spectral theory.

1　Introduction

P. R. Halmos said ([4] p.100), "one of the most important, most difficult, and most exasperating unsolved problems of operator theory is the problem of invariant subspace". The question is simple to state: does every bounded linear operator have a non-trivial invariant closed subspace?

The main motivations for the study of invariant subspaces come form interest in the structure of operators. The Jordan Space Decomposition Theorem for operators on finite dimensional spaces can be regarded as exhibiting operators as direct sum of their restrictions to certain invariant closed subspaces(cf.[6]).

For the convenience of readers, we first recall some of basic notions and facts.

Let X be a linear space, and let $X_1, X_2, ..., X_n$ be linear subspaces of X. If every $x \in X$ can be written as $x = x_1 + x_2 + ... + x_n$, where $x_k \in X_k, \ k =$

$1, 2, ..., n$, and this decomposition is unique, then X is called the direct sum of $X_1, X_2, ..., X_n$, and is written by $X = X_1 \oplus X_2 \oplus ... \oplus X_n$.

It is known that X is the direct sum of $X_1, X_2, ..., X_n$ if and only if $X = X_1 + X_2 + ... + X_n$ and $X_k \cap (\sum_{j \neq k} X_j) = \{0\}$ holds for every fixed $k = 1, 2, ..., n$.

Let X be a Banach space, and let $B(X)$ stand for the Banach algebra of all bounded linear operators on X.

2   Main Results

Theorem 2.1 *Let X be a complex Banach space. Let $T \in B(X)$ and $\sigma(T) = \sigma_1 \cup \sigma_2 \cup ... \cup \sigma_n$, $n = 2, 3, ...$, where $\sigma_1, \sigma_2, ..., \sigma_n$ are disjointwise non-empty closed set. Suppose that $L_1, L_2, ..., L_n$ are simple closed rectifiable curves in $\rho(T)$ such that σ_k is inside L_k for every $k = 1, 2, ..., n$, and that L_j is outside L_k for all $j \neq k$. If every $L_k(k = 1, 2, ..., n)$ is oriented in counterclockwise (that is, L_k is positively oriented), then we have*

(a) For every $k = 1, 2, ..., n$, the operator

$$P_k = \frac{1}{2\pi i} \int_{L_k} (z - T)^{-1} dz$$

is a bounded linear operator on X, and it is also a projection operator on X, that is, $P_k^2 = P_k$.

(b) If $S \in B(X)$ and $ST = TS$, then $SP_k = P_k S$ for every $k = 1, 2, ..., n$.

(c) For every $k = 1, 2, ..., n$, write $X_k = ran P_k$, then X_k is a hyperinvariant closed subspace for T, and

$$X_k = ker(I - P_k) = \{x | P_k x = x\},$$

where I denotes the identity operator on X.

(d) $\sigma(T|X_k) = \sigma_k$, $k = 1, 2, ..., n$.

(e) $X_k \neq \{0\}$, $X_k \neq X$, $k = 1, 2, ..., n$.

(f) $X = X_1 \oplus X_2 \oplus, ..., \oplus X_n$.

Proof. The main contribution of this proof is to show that the uniqueness of the space decomposition in Part (f) by means of Cauchy's Integral Theorem for operator-valued analytic functions on a multiply connected domain. It is easy to see that the proof of Part (a) to Part (e) is analogous to that in case $n = 2$. For a full discussion and proofs of these facts the reader is refered to [1], [6], [7]and so on. Hence the proof of part (a) to part (e) may be omitted. However, we still give a direct proof of $\sigma_k \subset \sigma(T|X_k)$ for the convenience of readers. For this end, we write $L = L_1 \cup L_2 \cup ... \cup L_n$, then $\sigma(T)$ is contained

inside L. Taking $f(z) \equiv 1$, it follows form the basic properties of the Riesz Functional Calculus that

$$I = \frac{1}{2\pi i} \int_L f(z)(z - T)^{-1}dz = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{L_k} (z - T)^{-1}dz = \sum_{k=1}^{n} P_k. \qquad (1)$$

We now show that $\sigma_k \subset \sigma(T|X_k), k = 1, 2, ..., n$. For this suppose that it were the case that there is a $z_0 \in \sigma_k$ but $z_0 \notin \sigma(T|X_k)$, then $z_0 \notin \sigma_j$ for every $j \neq k$. Thus by $\sigma(T|X_l) \subset \sigma_l, l = 1, 2, ..., n$ (the proof is left to the reader, cf. [6], p.31-p.32, or the others), we have $z_0 \notin \sigma(T|X_j)$ for every $j \neq k$. Hence $z_0 \in \rho(T|X_l)$, for every $l = 1, 2, ..., n$, this shows that $z_0 - T|X_l$ is invertible in the Banach algebra $B(X_l)$. Thus we have

$$(z_0 - T|X_l)^{-1}(z_0 - T)y = (z_0 - T)(z_0 - T|X_l)^{-1}y = y$$

for every $y \in X_l$ and every $l = 1, 2, ..., n$. We may define a bounded linear operator $\overline{S}$ on X by setting

$$\overline{S}x = (z_0 - T|X_1)^{-1}P_1x + (z_0 - T|X_2)^{-1}P_2x + ... + (z_0 - T|X_n)^{-1}P_nx$$

for every $x \in X$. It follows form $TP_l = P_lT$ and $P_lx \in X_l$ that

$$\overline{S}(z_0 - T)x = \sum_{l=1}^{n}(z_0 - T|X_l)^{-1}P_l(z_0 - T)x$$

$$= \sum_{l=1}^{n}(z_0 - T|X_l)^{-1}(z_0 - T)P_lx = \sum_{l=1}^{n} P_lx = x$$

and

$$(z_0 - T)\overline{S}x = \sum_{l=1}^{n}(z_0 - T)(z_0 - T|X_l)^{-1}P_lx = \sum_{l=1}^{n} P_lx = x$$

hold for every $x \in X$. Hence $\overline{S}(z_0 - T) = (z_0 - T)\overline{S} = I$, so that $z_0 \in \rho(T)$. This contradicts $z_0 \in \sigma_k \subset \sigma(T)$.

Part (f).We now prove that $X = X_1 \oplus X_2 \oplus ... \oplus X_n$.

It should be mentioned that of key importance will be Cauchy′s Integral Theorem for operator-valued analytic functions on a multiply connected domain which is used to show that the uniqueness of the space decomposition in case n=3,4,5,....

But first, we can see from (1) that the equality $x = P_1x + P_2x + ...P_nx$ holds for every $x \in X$, this implies

$$X = X_1 + X_2 + ... + X_n. \qquad (2)$$

It remains to prove the uniqueness of the space decomposition $X = X_1 + X_2 + ... + X_n$. For this it must be shown that the equality $X_k \cap (\sum_{j \neq k} X_j) = \{0\}$ holds for every fixed $k = 1, 2, ..., n$.

In fact, in case $n = 2$, if $x \in X_1 \cap X_2$, then by Part (c) we have $x \in X_1 = \{x | P_1 x = x\}$ and $x \in X_2 = \ker(I - P_2) = \ker P_1$. This implies $P_1 x = x$ and $P_1 x = 0$, and so that $x = 0$. It follows that $X_1 \cap X_2 = \{0\}$.

It is clear that the proof of the the uniqueness of the space decomposition can not be obtained by using induction from 2 to n.

In case $n = 3, 4, 5, ...$, suppose that $L_k^{'}$ is a simple closed rectifiable (positively oriented) curve such that $\bigcup_{j \neq k} L_j$ is contained inside $L_k^{'}$ (therefore $\bigcup_{j \neq k} \sigma_j$ is contained inside $L_k^{'}$), and L_k is outside $L_k^{'}$. Then we may define a bounded linear operator $P_k^{'}$ on X by setting

$$P_k^{'} = \frac{1}{2\pi i} \int_{L_k^{'}} (z - T)^{-1} dz.$$

Thus by Cauchy's Integral Theorem for operator-valued analytic functions on a multiply connected domain, we have

$$P_k^{'} = \frac{1}{2\pi i} \int_{L_k^{'}} (z - T)^{-1} dz = \frac{1}{2\pi i} \sum_{j \neq k} \int_{L_j} (z - T)^{-1} dz = \sum_{j \neq k} P_j.$$

Write $\sigma_k^{'} = \bigcup_{j \neq k} \sigma_j$, then we have $\sigma(T) = \sigma_k \cup \sigma_k^{'}$. It is clear that σ_k and $\sigma_k^{'}$ are disjoint non-empty closed sets and that $P_k^{'} = \sum_{j \neq k} P_j = I - P_k$. Write $X_k^{'} = \mathrm{ran} P_k^{'}$, then we have $X_k^{'} = \mathrm{ran} P_k^{'} = \sum_{j \neq k} P_j X = \sum_{j \neq k} X_j$. Thus by the result in case $n = 2$, we have

$$X_k \bigcap (\sum_{j \neq k} X_j) = X_k \bigcap X_k^{'} = \{0\}.$$

Consequently by (2) we obtain $X = X_1 \oplus X_2 \oplus ... \oplus X_n$. The proof is complete.

Corollary 2.2 *Let X be a complex Banach space, $T \in B(X)$. If $\sigma(T) = \sigma_1 \cup \sigma_2 \cup ... \cup \sigma_n$, $n = 2, 3, ...$, where $\sigma_1, \sigma_2, ..., \sigma_n$ are disjointwise non-empty closed sets, then T has non-trivial hyperinvariant invariant closed subspace $X_1, X_2, ..., X_n$ such that*

$$X = X_1 \oplus X_2 \oplus ... \oplus X_n, \quad \sigma(T | X_k) = \sigma_k, \quad k = 1, 2, ..., n,$$

where $X_k = ker(I - P_k)$, while I and $P_k | X_k$ $(k = 1, 2, ..., n)$ are the identity operators on X and X_k respectively.

Corollary 2.3 *(Jordan space Decomposition Theorem, see [1] and so on)Let X be a finite dimensional complex linear space, let $T \in B(X)$. If $\lambda_1, \lambda_2, ..., \lambda_n (n \geq 2)$ are all eigenvalues of T, and r_k is the multiplicity of eigenvalue $\lambda_k (k = 1, 2, ..., m)$, then T has non-trivial hyperinvariant closed subspaces $X_1, X_2, ..., X_n$ such that*

$$X = X_1 \oplus X_2 \oplus, ..., \oplus X_n, \quad \sigma_p(T | X_k) = \{\lambda_k\}, \quad k = 1, 2, ..., n,$$

where $X_k = ker(\lambda_k - T)^{r_k}$, and $\{\lambda_k\}$ is a singleton.

3 Remarks

Remark 1. Corollary 2.3 is just the Jordan Space Decomposition Theorem. In case $n = 2$, Corollary 2.2 is the Riesz Space Decomposition Theorem. Since Jordan′s theorem and Riesz′s theorem are two famous results in operator theory, they often appear in many books, and are even detailed in the same book. For example, Jordan Space Decomposition Theorem is detailed in Chapter 1 of [1], at the same time Riesz Space Decomposition Theorem is also detailed in Chapter 2 of the same book that is a nice research book on operator theory and the invariant subspace problem.

It is clear that by Theorem 2.1 one has Corollary 2.2. On the other hand, by the spectral theory, if X_k is a finite dimensional space, then $\sigma(T|X_k) = \sigma_p(T|X_k)$, $k = 1, 2, ..., n$. Form this it is easy to see that Riesz Space Decomposition Theorem is a corollary of Corollary 2.2. Thus after we obtain Theorem 2.1, The discussion of two theorem above may be omitted, while the result of Jordan Space Decomposition Theorem and Riesz Space Decomposition Theorem will be obtained as corollaries of Theorem 2.1.

For example, one may modify Chapter 2 of [1], then Chapter 1 of [1] is omitted.

Remark 2. It is also worth mentioning that in case n=3,4,5,... of Theorem 2.1 we prove the uniqueness of the space decomposition by means of Cauchy's Integral Theorem for operator-valued analytic functions on a multiply connected domain, while in Jordan Space Decomposition Theorem and Riesz Space Decomposition Theorem ($n = 2$) one prove the uniqueness of the space decomposition by means of the algebraic method (see [1], [6] and so on; see also the proof of Theorem 1 of this paper in case $n = 2$), and it seem's to me that it is impossible that the uniqueness of the infinite dimensional space decomposition is shown by means of the algebra method in case $n = 3, 4, 5....$

Remark 3. We conjecture that Theorem 2.1 can be extend to $n = \infty$. It is easy to see that to prove the conjecture it also suffices to deal with $X = X_1 \oplus X_2 \oplus ... \oplus X_n \oplus ...$,while the others are also analogous to that in case $n = 2$. The conjecture is interesting, for example, if X is a infinite dimensional complex Banach space, and $\sigma(T) = \{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$,here $\lambda_n \neq \lambda_m, n \neq m$ (a compact operator on a infinite dimensional complex Banach space is one of them).

References

[1] B. Beauzamy, Introduction to Operator Theory and Invariant Subspaces, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1988.

[2] B. Beauzamy, Sous-espaces invariants de type fonctionnel dans les Espace de Banach, Acta Math., 144 (1981), 27-64.

[3] B. Beauzamy, Un opérateur sans sous-espace invariant non-trival: simplification de l'exemple de P. Enflo. Integral Equations and operator Theory, 8 (1985), 314-384.

[4] P. R. Halmos, A Hilbert Space Problem Book, second ed., Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[5] H. Radjavi and P. Rosenthal, On invariant subspaces and reflexive algebras. Amer. J. Math., 91 (1969), 683-692.

[6] H. Radjavi and P. Rosenthal, Invariant Subspaces, Springer-Verlag, second ed., New York, 2003.

[7] D.Xia and S.Yan, An Introduction to Functions of a Real Variable and Applied Functional Analysis, Shanghai Science and Technology, Shanghai, 1987.