

# On the Periodic Solutions of Some Rational Difference Systems I

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## Abstract

In this paper, we focus on the system of rational difference equations

$$x_{n+1} = \frac{\beta y_{n-2}}{-\beta + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{\beta x_{n-2}}{-\beta + x_{n-2}y_{n-1}x_n},$$
$$z_{n+1} = \frac{\beta x_{n-2} + \beta y_{n-2}}{-\beta + x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \dots,$$

where the initial conditions are arbitrary real numbers and  $\beta \neq 0$ . We investigate the periodic solutions of the system.

**Mathematics Subject Classification:** 39A10

**Keywords:** rational difference equation, periodic solution

## 1 Introduction

In 2011, Kurbanli [2] studied the behavior of solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}, \quad n = 0, 1, \dots,$$

where the initial conditions are arbitrary real numbers.

In 2012, Keying et al. [1] studied the behavior of the solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{1}{x_n z_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions are nonzero real numbers.

In 2013, Özkan and Kurbanli [3] studied the periodic solutions of the system of rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n},$$

$$z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \dots,$$

where the initial conditions are arbitrary real numbers.

## 2 Results

**Theorem 2.1.** *Let  $\beta$  be a nonzero real number, and let  $y_0 = a$ ,  $y_{-1} = b$ ,  $y_{-2} = c$ ,  $x_0 = d$ ,  $x_{-1} = e$ ,  $x_{-2} = f$  be arbitrary real numbers such that  $fbd \neq \beta \neq cea$ . Let  $\{x_n, y_n, z_n\}$  be a solution of the system*

$$x_{n+1} = \frac{\beta y_{n-2}}{-\beta + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{\beta x_{n-2}}{-\beta + x_{n-2}y_{n-1}x_n},$$

$$z_{n+1} = \frac{\beta x_{n-2} + \beta y_{n-2}}{-\beta + x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, \dots$$

*Then all six-period solutions of the system are as follows:*

$$x_{6n+1} = \frac{\beta c}{cea - \beta}, \quad y_{6n+1} = \frac{\beta f}{fbd - \beta}, \quad z_{6n+1} = \frac{\beta f + \beta c}{fbd - \beta},$$

$$x_{6n+2} = \frac{b(fbd - \beta)}{\beta}, \quad y_{6n+2} = \frac{e(cea - \beta)}{\beta}, \quad z_{6n+2} = \frac{(e + b)(cea - \beta)}{\beta},$$

$$x_{6n+3} = \frac{\beta a}{cea - \beta}, \quad y_{6n+3} = \frac{\beta d}{fbd - \beta}, \quad z_{6n+3} = \frac{\beta d + \beta a}{fbd - \beta},$$

$$x_{6n+4} = f, \quad y_{6n+4} = c, \quad z_{6n+4} = \frac{c(fbd - \beta) + f(cea - \beta)}{fbd - \beta},$$

$$x_{6n+5} = e, \quad y_{6n+5} = b, \quad z_{6n+5} = \frac{b(fbd - \beta) + e(cea - \beta)}{fbd - \beta},$$

$$x_{6n+6} = d, \quad y_{6n+6} = a, \quad z_{6n+6} = \frac{a(fbd - \beta) + d(cea - \beta)}{fbd - \beta},$$

where  $n = 0, 1, \dots$

*Proof.* We note that  $x_1 = \frac{\beta c}{cea - \beta}$ ,  $y_1 = \frac{\beta f}{fbd - \beta}$ ,  $z_1 = \frac{\beta f + \beta c}{fbd - \beta}$ ,

$$x_2 = \frac{\beta y_{-1}}{-\beta + y_{-1}x_0y_1} = \frac{\beta b}{-\beta + bd \left( \frac{\beta f}{fbd - \beta} \right)} = \frac{b(fbd - \beta)}{\beta},$$

$$y_2 = \frac{\beta x_{-1}}{-\beta + x_{-1}y_0x_1} = \frac{\beta e}{-\beta + ea \left( \frac{\beta c}{cea - \beta} \right)} = \frac{e(cea - \beta)}{\beta},$$

$$z_2 = \frac{\beta x_{-1} + \beta y_{-1}}{-\beta + x_{-1}y_0x_1} = \frac{\beta e + \beta b}{-\beta + ea \left( \frac{\beta c}{cea - \beta} \right)} = \frac{(e + b)(cea - \beta)}{\beta},$$

$$x_3 = \frac{\beta y_0}{-\beta + y_0x_1y_2} = \frac{\beta a}{-\beta + a \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right)} = \frac{\beta a}{cea - \beta},$$

$$y_3 = \frac{\beta x_0}{-\beta + x_0y_1x_2} = \frac{\beta d}{-\beta + d \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right)} = \frac{\beta d}{fbd - \beta},$$

$$z_3 = \frac{\beta x_0 + \beta y_0}{-\beta + x_0y_1x_2} = \frac{\beta d + \beta a}{-\beta + d \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right)} = \frac{\beta d + \beta a}{fbd - \beta},$$

$$x_4 = \frac{\beta y_1}{-\beta + y_1x_2y_3} = \frac{\beta \left( \frac{\beta f}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right)} = f,$$

$$y_4 = \frac{\beta x_1}{-\beta + x_1y_2x_3} = \frac{\beta \left( \frac{\beta c}{cea - \beta} \right)}{-\beta + \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right)} = c,$$

$$\begin{aligned} z_4 &= \frac{\beta x_1 + \beta y_1}{-\beta + x_1y_2x_3} \\ &= \frac{\beta \left( \frac{\beta c}{cea - \beta} \right) + \beta \left( \frac{\beta f}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right)} \\ &= \frac{c(fbd - \beta) + f(cea - \beta)}{fbd - \beta}, \end{aligned}$$

$$x_5 = \frac{\beta y_2}{-\beta + y_2 x_3 y_4} = \frac{\beta \left( \frac{e(cea - \beta)}{\beta} \right)}{-\beta + \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right) c} = e,$$

$$y_5 = \frac{\beta x_2}{-\beta + x_2 y_3 x_4} = \frac{\beta \left( \frac{b(fbd - \beta)}{\beta} \right)}{-\beta + \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right) f} = b,$$

$$\begin{aligned} z_5 &= \frac{\beta x_2 + \beta y_2}{-\beta + x_2 y_3 x_4} \\ &= \frac{\beta \left( \frac{b(fbd - \beta)}{\beta} \right) + \beta \left( \frac{e(cea - \beta)}{\beta} \right)}{-\beta + \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right) f} \\ &= \frac{b(fbd - \beta) + e(cea - \beta)}{fbd - \beta}, \end{aligned}$$

$$x_6 = \frac{\beta y_3}{-\beta + y_3 x_4 y_5} = \frac{\beta \left( \frac{\beta d}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta d}{fbd - \beta} \right) fb} = d,$$

$$y_6 = \frac{\beta x_3}{-\beta + x_3 y_4 x_5} = \frac{\beta \left( \frac{\beta a}{cea - \beta} \right)}{-\beta + \left( \frac{\beta a}{cea - \beta} \right) ce} = a,$$

$$\begin{aligned} z_6 &= \frac{\beta x_3 + \beta y_3}{-\beta + x_3 y_4 x_5} \\ &= \frac{\beta \left( \frac{\beta a}{cea - \beta} \right) + \beta \left( \frac{\beta d}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta a}{cea - \beta} \right) ce} \\ &= \frac{a(fbd - \beta) + d(cea - \beta)}{fbd - \beta}. \end{aligned}$$

Then we obtain that

$$\begin{aligned} x_7 &= \frac{\beta y_4}{-\beta + y_4 x_5 y_6} = \frac{\beta c}{cea - \beta} = x_1, \\ y_7 &= \frac{\beta x_4}{-\beta + x_4 y_5 x_6} = \frac{\beta f}{fbd - \beta} = y_1, \\ z_7 &= \frac{\beta x_4 + \beta y_4}{-\beta + x_4 y_5 x_6} = \frac{\beta f + \beta c}{fbd - \beta} = z_1, \end{aligned}$$

$$\begin{aligned} x_8 &= \frac{\beta y_5}{-\beta + y_5 x_6 y_7} = \frac{\beta b}{-\beta + bd \left( \frac{\beta f}{fbd - \beta} \right)} = \frac{b(fbd - \beta)}{\beta} = x_2, \\ y_8 &= \frac{\beta x_5}{-\beta + x_5 y_6 x_7} = \frac{\beta e}{-\beta + ea \left( \frac{\beta c}{cea - \beta} \right)} = \frac{e(cea - \beta)}{\beta} = y_2, \\ z_8 &= \frac{\beta x_5 + \beta y_5}{-\beta + x_5 y_6 x_7} = \frac{\beta e + \beta b}{-\beta + ea \left( \frac{\beta c}{cea - \beta} \right)} = \frac{(e + b)(cea - \beta)}{\beta} = z_2, \end{aligned}$$

$$\begin{aligned} x_9 &= \frac{\beta y_6}{-\beta + y_6 x_7 y_8} = \frac{\beta a}{-\beta + a \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right)} = \frac{\beta a}{cea - \beta} = x_3, \\ y_9 &= \frac{\beta x_6}{-\beta + x_6 y_7 x_8} = \frac{\beta d}{-\beta + d \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right)} = \frac{\beta d}{fbd - \beta} = y_3, \\ z_9 &= \frac{\beta x_6 + \beta y_6}{-\beta + x_6 y_7 x_8} = \frac{\beta d + \beta a}{-\beta + d \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right)} = \frac{\beta d + \beta a}{fbd - \beta} = z_3, \end{aligned}$$

$$\begin{aligned} x_{10} &= \frac{\beta y_7}{-\beta + y_7 x_8 y_9} = \frac{\beta \left( \frac{\beta f}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta f}{fbd - \beta} \right) \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right)} = f = x_4, \\ y_{10} &= \frac{\beta x_7}{-\beta + x_7 y_8 x_9} = \frac{\beta \left( \frac{\beta c}{cea - \beta} \right)}{-\beta + \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right)} = c = y_4, \end{aligned}$$

$$\begin{aligned}
z_{10} &= \frac{\beta x_7 + \beta y_7}{-\beta + x_7 y_8 x_9} \\
&= \frac{\beta \left( \frac{\beta c}{cea - \beta} \right) + \beta \left( \frac{\beta f}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta c}{cea - \beta} \right) \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right)} \\
&= \frac{c(fbd - \beta) + f(cea - \beta)}{fbd - \beta} \\
&= z_4,
\end{aligned}$$

$$\begin{aligned}
x_{11} &= \frac{\beta y_8}{-\beta + y_8 x_9 y_{10}} = \frac{\beta \left( \frac{e(cea - \beta)}{\beta} \right)}{-\beta + \left( \frac{e(cea - \beta)}{\beta} \right) \left( \frac{\beta a}{cea - \beta} \right) c} = e = x_5, \\
y_{11} &= \frac{\beta x_8}{-\beta + x_8 y_9 x_{10}} = \frac{\beta \left( \frac{b(fbd - \beta)}{\beta} \right)}{-\beta + \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right) f} = b = y_5,
\end{aligned}$$

$$\begin{aligned}
z_{11} &= \frac{\beta x_8 + \beta y_8}{-\beta + x_8 y_9 x_{10}} \\
&= \frac{\beta \left( \frac{b(fbd - \beta)}{\beta} \right) + \beta \left( \frac{e(cea - \beta)}{\beta} \right)}{-\beta + \left( \frac{b(fbd - \beta)}{\beta} \right) \left( \frac{\beta d}{fbd - \beta} \right) f} \\
&= \frac{b(fbd - \beta) + e(cea - \beta)}{fbd - \beta} \\
&= z_5,
\end{aligned}$$

$$\begin{aligned}
x_{12} &= \frac{\beta y_9}{-\beta + y_9 x_{10} y_{11}} = \frac{\beta \left( \frac{\beta d}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta d}{fbd - \beta} \right) fb} = d = x_6, \\
y_{12} &= \frac{\beta x_9}{-\beta + x_9 y_{10} x_{11}} = \frac{\beta \left( \frac{\beta a}{cea - \beta} \right)}{-\beta + \left( \frac{\beta a}{cea - \beta} \right) ce} = a = y_6,
\end{aligned}$$

$$\begin{aligned}
 z_{12} &= \frac{\beta x_9 + \beta y_9}{-\beta + x_9 y_{10} x_{11}} \\
 &= \frac{\beta \left( \frac{\beta a}{cea - \beta} \right) + \beta \left( \frac{\beta d}{fbd - \beta} \right)}{-\beta + \left( \frac{\beta a}{cea - \beta} \right) ce} \\
 &= \frac{a(fbd - \beta) + d(cea - \beta)}{fbd - \beta} \\
 &= z_6.
 \end{aligned}$$

Next, we let  $m \in \mathbb{N}$ . Suppose that  $x_{6m+1} = x_1, y_{6m+1} = y_1, z_{6m+1} = z_1, x_{6m+2} = x_2, y_{6m+2} = y_2, z_{6m+2} = z_2, x_{6m+3} = x_3, y_{6m+3} = y_3, z_{6m+3} = z_3, x_{6m+4} = x_4, y_{6m+4} = y_4, z_{6m+4} = z_4, x_{6m+5} = x_5, y_{6m+5} = y_5, z_{6m+5} = z_5, x_{6m+6} = x_6, y_{6m+6} = y_6$  and  $z_{6m+6} = z_6$ . Then

$$\begin{aligned}
 x_{6m+7} &= \frac{\beta y_{6m+4}}{-\beta + y_{6m+4} x_{6m+5} y_{6m+6}} = \frac{\beta y_4}{-\beta + y_4 x_5 y_6} = x_7 = x_1, \\
 y_{6m+7} &= \frac{\beta x_{6m+4}}{-\beta + x_{6m+4} y_{6m+5} x_{6m+6}} = \frac{\beta x_4}{-\beta + x_4 y_5 x_6} = y_7 = y_1, \\
 z_{6m+7} &= \frac{\beta x_{6m+4} + \beta y_{6m+4}}{-\beta + x_{6m+4} y_{6m+5} x_{6m+6}} = \frac{\beta x_4 + \beta y_4}{-\beta + x_4 y_5 x_6} = z_7 = z_1, \\
 x_{6m+8} &= \frac{\beta y_{6m+5}}{-\beta + y_{6m+5} x_{6m+6} y_{6m+7}} = \frac{\beta y_5}{-\beta + y_5 x_6 y_7} = x_8 = x_2, \\
 y_{6m+8} &= \frac{\beta x_{6m+5}}{-\beta + x_{6m+5} y_{6m+6} x_{6m+7}} = \frac{\beta x_5}{-\beta + x_5 y_6 x_7} = y_8 = y_2, \\
 z_{6m+8} &= \frac{\beta x_{6m+5} + \beta y_{6m+5}}{-\beta + x_{6m+5} y_{6m+6} x_{6m+7}} = \frac{\beta x_5 + \beta y_5}{-\beta + x_5 y_6 x_7} = z_8 = z_2, \\
 x_{6m+9} &= \frac{\beta y_{6m+6}}{-\beta + y_{6m+6} x_{6m+7} y_{6m+8}} = \frac{\beta y_6}{-\beta + y_6 x_7 y_8} = x_9 = x_3, \\
 y_{6m+9} &= \frac{\beta x_{6m+6}}{-\beta + x_{6m+6} y_{6m+7} x_{6m+8}} = \frac{\beta x_6}{-\beta + x_6 y_7 x_8} = y_9 = y_3, \\
 z_{6m+9} &= \frac{\beta x_{6m+6} + \beta y_{6m+6}}{-\beta + x_{6m+6} y_{6m+7} x_{6m+8}} = \frac{\beta x_6 + \beta y_6}{-\beta + x_6 y_7 x_8} = z_9 = z_3, \\
 x_{6m+10} &= \frac{\beta y_{6m+7}}{-\beta + y_{6m+7} x_{6m+8} y_{6m+9}} = \frac{\beta y_7}{-\beta + y_7 x_8 y_9} = x_{10} = x_4, \\
 y_{6m+10} &= \frac{\beta x_{6m+7}}{-\beta + x_{6m+7} y_{6m+8} x_{6m+9}} = \frac{\beta x_7}{-\beta + x_7 y_8 x_9} = y_{10} = y_4, \\
 z_{6m+10} &= \frac{\beta x_{6m+7} + \beta y_{6m+7}}{-\beta + x_{6m+7} y_{6m+8} x_{6m+9}} = \frac{\beta x_7 + \beta y_7}{-\beta + x_7 y_8 x_9} = z_{10} = z_4,
 \end{aligned}$$

$$\begin{aligned}
x_{6m+11} &= \frac{\beta y_{6m+8}}{-\beta + y_{6m+8}x_{6m+9}y_{6m+10}} = \frac{\beta y_8}{-\beta + y_8x_9y_{10}} = x_{11} = x_5, \\
y_{6m+11} &= \frac{\beta x_{6m+8}}{-\beta + x_{6m+8}y_{6m+9}x_{6m+10}} = \frac{\beta x_8}{-\beta + x_8y_9x_{10}} = y_{11} = y_5, \\
z_{6m+11} &= \frac{\beta x_{6m+8} + \beta y_{6m+8}}{-\beta + x_{6m+8}y_{6m+9}x_{6m+10}} = \frac{\beta x_8 + \beta y_8}{-\beta + x_8y_9x_{10}} = z_{11} = z_5, \\
\\
x_{6m+12} &= \frac{\beta y_{6m+9}}{-\beta + y_{6m+9}x_{6m+10}y_{6m+11}} = \frac{\beta y_9}{-\beta + y_9x_{10}y_{11}} = x_{12} = x_6, \\
y_{6m+12} &= \frac{\beta x_{6m+9}}{-\beta + x_{6m+9}y_{6m+10}x_{6m+11}} = \frac{\beta x_9}{-\beta + x_9y_{10}x_{11}} = y_{12} = y_6, \\
z_{6m+12} &= \frac{\beta x_{6m+9} + \beta y_{6m+9}}{-\beta + x_{6m+9}y_{6m+10}x_{6m+11}} = \frac{\beta x_9 + \beta y_9}{-\beta + x_9y_{10}x_{11}} = z_{12} = z_6.
\end{aligned}$$

By the mathematical induction, this proof is completed.  $\square$

## References

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