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On the oscillatory behavior of solutions of a class of nonlinear fractional difference equations

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Abstract

In this paper, we study the oscillatory behavior of solutions of a class of Caputo fractional difference equations. Some new criteria for the oscillation of fractional difference equations is established.

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1 Introduction

Accompanied with the development of the theory on fractional differential equations, fractional difference equations have also been studied more intensively of late. Some properties and inequalities of the fractional difference calculus are discussed in [1−7], the existence and asymptotic stability of the solutions for fractional difference equations are investigated in [8−10], and the boundary value problems of fractional difference equations are considered in [11−13]. But, the oscillation results for fractional difference equations are scarce.

In this paper we investigate the oscillation of fractional difference equations:

$$\begin{cases} \Delta_C^v x(t) = e(t+v-1) + f(t+v-1, x(t+v-1)), t \in \mathbb{N}_0; \\ x(v-2) = x_0, \Delta x(v-2) = x_1, \end{cases} \tag{1}$$

where Δ_C^v is a Caputo fractional difference operator, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $1 < v < 2$, x_0 and x_1 are real constants, $f : \mathbb{N}_{v-1} \times \mathbb{R} \to \mathbb{R}^+$ is continuous, and $e : [v-1, +\infty) \to \mathbb{R}$ is continuous.

2 Preliminary notes

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 ([3,4]) *Let $v > 0$. The vth fractional sum of $f : \mathbb{N}_a \to \mathbb{R}$ is defined by*

$$\Delta^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} f(s), t \in \mathbb{N}_{a+v},$$

where $t^{(v)} = \Gamma(t+1)/\Gamma(t-v+1)$.

Definition 2.2 ([1]) *Let $\mu > 0$ and $n-1 < \mu < n$, where n denotes a positive integer and $n = \lceil \mu \rceil$, $\lceil \cdot \rceil$ ceiling of number. Set $v = n - \mu$. The μth fractional Caputo difference operator is defined as*

$$\Delta_C^{\mu} f(t) = \Delta^{-v}(\Delta^n f(t)) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} \Delta^n f(s), t \in \mathbb{N}_{a+v},$$

where Δ^n is the nth order forward difference operator; the fractional Caputo like difference Δ_C^{μ} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a-\mu}$.

Lemma 2.3 ([2,13]) *Assume that $\mu > 0$ and f is defined on $\mathbb{N}_a$. Then,*

$$\Delta^{-\mu}\Delta_C^{\mu} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) = f(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{(n-1)},$$

where n is the smallest integer greater than or equal to μ, $c_i \in \mathbb{R}, i = 1, 2, \ldots, n-1$.

Lemma 2.4 [14] *If X and Y are nonnegative,then*

$$X^{\lambda} - (1-\lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0, 0 < \lambda < 1,$$

where equality hold if and only if $X = Y$.

Lemma 2.5 *Assume that any solution x(t) of the fractional difference equations (1) exists on the interval $\mathbb{N}_{v-2}$. Then the equations (1) is equivalent to the equation*

$$x(t) = x_0 - x_1(v-2) + x_1 t + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)} [e(s+v-1) + f(s+v-1, x(s+v-1))]. \quad (2)$$

Proof By Lemma 2.3, we have

$$x(t) = \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)}[e(s+v-1)+f(s+v-1, x(s+v-1))]+c_0+c_1 t. \quad (3)$$

Then, we obtain

$$\Delta x(t) = \frac{1}{\Gamma(v-1)} \sum_{s=0}^{t-v+1} (t-s-1)^{(v-1)}[e(s+v-1)+f(s+v-1, x(s+v-1))]+c_1.$$

In view of $x(v-2) = x_0$ and $\Delta x(v-2) = x_1$, we have $c_0 + c_1(v-2) = x_0$ and $c_1 = x_1$. Then, $c_0 = x_0 - x_1(v-2)$ and $c_1 = x_1$. Substituting the values of c_0 and c_1 into (3), we obtain (2). The proof is complete.

3 Main results

In this section, we state and prove our main results.

We assume that there exists a continuous function $h : [v-1, +\infty) \to (0, +\infty)$ and real number λ and with $0 < \lambda \leq 1$ such that

$$0 \leq x f(t,x) \leq h(t)|x|^{\lambda+1}, \quad x \neq 0, t \in \mathbb{N}_{v-1}. \quad (4)$$

In what follows, we let

$$g_{\pm}(t) = \frac{1}{\Gamma(v)} \sum_{s=t_1-v+1}^{t-v} (t-s-1)^{(v-1)}[\pm e(s+v-1)+(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} m^{\frac{\lambda}{1-\lambda}}(s+v-1) h^{\frac{1}{1-\lambda}}(s+v-1)] \quad (5)$$

where $0 < \lambda < 1, t \geq t_1$, for some $t_1 \in \mathbb{N}_{v-1}$, $m : [v-1, +\infty) \to (0, +\infty)$ is a given continuous function.

Now we give sufficient conditions under which a nonoscillatory solution x of the equations (1) satisfies $\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty$.

Theorem 3.1 *Let $0 < \lambda < 1, 1 < v < 2$ and suppose*

$$\liminf_{t\to\infty} \frac{g_{\pm}(t)}{t} > -\infty \quad \textit{and} \quad \limsup_{t\to\infty} \frac{g_{\pm}(t)}{t} < \infty, \quad (6)$$

and

$$\sum_{s=t_1}^{\infty} m(s)s < \infty, \quad t, t_1 \in \mathbb{N}_{v-1}. \quad (7)$$

If $x(t)$ is a nonoscillatory solution of the equations (1), then

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \quad (8)$$

Proof Let $x(t)$ be a nonoscillatory solution of the equations (1). We may assume that $x(t) > 0$, for all $t > t_1, t, t_1 \in \mathbb{N}_{v-1}$. Let $F(t) = e(t)+f(t,x(t)), C_0 = x_0 - x_1(v-2)$. In view of (2) we may then write

$$\begin{aligned} x(t) &= C_0 + x_1 t + \frac{1}{\Gamma(v)}\sum_{s=0}^{t_1-v}(t-s-1)^{(v-1)}F(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}F(s+v-1) \\ &\leq C_0 + x_1 t + \frac{1}{\Gamma(v)}\sum_{s=0}^{t_1-v}(t-s-1)^{(v-1)}F(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}e(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}[h(s+v-1)x^{\lambda}(s+v-1) - m(s+v-1)x(s+v-1)] \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}m(s+v-1)x(s+v-1) \end{aligned}$$

Applying Lemma2.4 to $h(s)x^{\lambda}(s) - m(s)x(s)$ with

$$X = h^{\frac{1}{\lambda}}x \quad and \quad Y = (\frac{1}{\lambda}mh^{-\frac{1}{\lambda}})^{\frac{1}{\lambda-1}},$$

we have

$$h(s)x^{\lambda}(s) - m(s)x(s) \leq (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}m^{\frac{\lambda}{1-\lambda}}h^{\frac{1}{1-\lambda}},$$

and so

$$\begin{aligned} x(t) &\leq C_0 + x_1 t + \frac{1}{\Gamma(v)}\sum_{s=0}^{t_1-v}t(t-s-1)^{(v-2)}F(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}e(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}m^{\frac{\lambda}{1-\lambda}}(s+v-1)h^{\frac{1}{1-\lambda}}(s+v-1) \\ &+ \frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v}(t-s-1)^{(v-1)}m(s+v-1)x(s+v-1) \\ &\leq C_0 + x_1 t + \frac{1}{\Gamma(v)}\sum_{s=0}^{t_1-v}t(t-s-1)^{(v-2)}|F(s+v-1)| + g_+(t) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(v)} \sum_{s=t_1-v+1}^{t-v} t(t-s-1)^{(v-2)} m(s+v-1) x(s+v-1) \\
&\leq \quad Ct + \frac{t}{\Gamma(v)} \sum_{s=t_1-v+1}^{t-v} (t-s-1)^{(v-2)} m(s+v-1) x(s+v-1),
\end{aligned}
$$

or

$$
\frac{x(t)}{t} := z(t) \leq 1+C+\frac{1}{\Gamma(v)} \sum_{s=t_1-v+1}^{t-v} (t-s-1)^{(v-2)} m(s+v-1) \cdot (s+v-1) \cdot z(s+v-1),
$$

where C is the upper bound of the function

$$
\frac{C_0}{t} + x_1 + \frac{1}{\Gamma(v)} \sum_{s=0}^{t_1-v} (t-s-1)^{(v-2)} |F(s+v-1)| + \frac{g_+(t)}{t}.
$$

According to $\Delta_s(t-s-1)^{(v-2)} = -(v-2)(t-s-2)^{(v-3)} > 0, t \in \mathbb{N}_{v-1}, s \in \{0, 1, 2, \ldots, t-v\}$, we have $\Delta_s(t-s-1)^{(v-2)}$ is nondecreasing for s, therefore,

$$
(t-s-1)^{(v-2)} < (t-t+v-1)^{(v-2)} = \Gamma(v).
$$

Hence, we have

$$
\begin{aligned}
z(t) \quad &\leq \quad 1 + C + \sum_{s=t_1-v+1}^{t-v} m(s+v-1) \cdot (s+v-1) \cdot z(s+v-1) \\
&= \quad 1 + C + \sum_{s=t_1}^{t-1} m(s) \cdot s \cdot z(s) \\
&\leq \quad 1 + C + \sum_{s=t_1}^{t} m(s) \cdot s \cdot z(s).
\end{aligned}
$$

Applying dispersed Bellman inequality and (7), we obtain

$$
z(t) \leq (1+C) e^{\sum_{s=t_1}^{t} m(s)s} < \infty.
$$

We conclude that

$$
\limsup_{t \to \infty} \frac{x(t)}{t} < \infty. \tag{9}
$$

If $x(t)$ is eventually negative, we can set $y = -x$ to see that y satisfies the equations (1) with $e(t)$ be replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. It follows in a similar manner that

$$
\limsup_{t \to \infty} \frac{-x(t)}{t} < \infty. \tag{10}
$$

From (9) and (10) we get (8). The proof is complete.

Theorem 3.2 *Let $0 < \lambda < 1$, the conditions(4),(6)and(7)hold. If for every constant $M, 0 < M < 1$*

$$\limsup_{t\to\infty}[Mt + g_-(t)] = \infty, \tag{11}$$

and

$$\liminf_{t\to\infty}[Mt + g_+(t)] = -\infty, \tag{12}$$

then the equations (1) is oscillatory.

Proof Let $x(t)$ be an onoscillatory solution of the equations (1). We may assume that $x(t) > 0$ for all $t \geq t_1$, for some $t_1 \in \mathbb{N}_{v-1}$. Proceeding similarly to the proof of Theorem 3.1, we obtain

$$x(t) \leq C_0 + x_1 t + \frac{1}{\Gamma(v)}\sum_{s=0}^{t_1-v}(t_1 - s - 1)^{(v-1)}|F(s + v - 1)| + g_+(t)$$

$$+\frac{1}{\Gamma(v)}\sum_{s=t_1-v+1}^{t-v} t(t - s - 1)^{(v-2)} m(s + v - 1)(s + v - 1)\frac{x(s + v - 1)}{(s + v - 1)} \tag{13}$$

Clearly, the conclusion of Theorem 3.1 holds. This together with(7) imply that the second sum on the right hand side of (13) is bounded and hence one can easily find

$$x(t) \leq M_0 + M_1 t + g_+(t) \tag{14}$$

where M_0 and M_1 (dependon t_1) are positive constants. Note that we make $M_1 < 1$ possible by increasing the size of t_1 .

Finally, taking lim inf in (14) as $t \to \infty$ and using (12) result in a contradiction with the fact that $x(t)$ is eventually positive.

If $x(t)$ is eventually negative, we can set $y = -x$ to see that y satisfies the equations (1) with $e(t)$ be replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. The proof of this case is the same as above and hence is omitted. This completes the proof of the theorem.

Similar to the sublinear case, one can easily prove the following results.

Theorem 3.3 *Let $\lambda = 1$, conditions (4) and (7) hold with $h(t) = m(t)$. Suppose*

$$\liminf_{t\to\infty}\frac{\sum_{s=0}^{t-v}(t - s - 1)^{(v-1)}e(s + v - 1)}{t} > -\infty \tag{15}$$

and

$$\limsup_{t\to\infty}\frac{\sum_{s=0}^{t-v}(t - s - 1)^{(v-1)}e(s + v - 1)}{t} < +\infty. \tag{16}$$

Then every nonoscillatory solution of the equations (1) satisfies (8).

Theorem 3.4 *Let $\lambda = 1$, conditions (4), (7), (15) and (16) hold with $h(t) = m(t)$. If for every constant $M, 0 < M < 1$*

$$\limsup_{t\to\infty}[Mt + \sum_{s=0}^{t-v}(t-s-1)^{(v-1)}e(s+v-1)] = \infty,$$

and

$$\liminf_{t\to\infty}[Mt + \sum_{s=0}^{t-v}(t-s-1)^{(v-1)}e(s+v-1)] = -\infty,$$

then the equations (1) is oscillatory.

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