On the New Skew distributions using Azzalini's Formula

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Abstract

We use the skew distribution generation procedure proposed by Azzalini [*Scand. J. Stat.*, **12**, 171-178, 1985] to create two new probability distribution functions. These models make use of Eirado-Rathie distribution ,see Eirado and Rathie [Statistics Research Letters, **3**, 17-22, 2014] and Rathie-Swamee generalized logistic distribution, see Rathie and Swamee [University of Brasilia: Brasilia, Brazil, 2006]. Expressions for the moments about origin are derived. Graphical illustrations are also provided. Applications with unimodal and bimodal data are given to illustrate the applicability of the results derived in this paper. The applications include the analysis of the following data sets: (a) spending on public education in various countries in 2003; (b) total expenditure on health in 2009 in various countries, (c) waiting time between eruptions of the Old Faithful Geyser in the Yellow Stone National Park, Wyoming, USA and (d) pH concentration.

Mathematics Subject Classification: 60E05, 62B15, 33C60, 60E10.

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1   Introduction

The skew symmetric models have been considered by several researchers. Skew normal distribution is a classical example. Abtahi *et al.* [1] constructed skew student-*t* and skew Cauchy distributions. Recently, Rathie *et al.* [9, 10, 11] introduced a system of univariate skew distributions by utilizing Rathie and Swamee [12] generalized logistic distribution. For certain values of the parameters, this skew distribution approximates nicely the skew normal distribution. In this paper we use Azzalini's formula to generate new asymmetric distributions by using the Rathie-Swamee generalized Logistic and Eirado-Rathie distribution (Eirado and Rathie, [4]). The distributions obtained are Skew Generalized Logistic - Eirado-Rathie ($h_1(x)$) and Skew Eirado-Rathie - Generalized Logistic ($h_2(x)$). These models are bimodal for certain values of the parameters, it is important to note that the values of the parameters determining uni/bi modal shapes are yet to be investigated.

We apply these distributions to four real data sets (expenditure on education, expenditure on health, waiting time between eruptions of the Old Faithful Geyser and pH concentration). We compare the fit of the distributions between them, the results show that: (1) in general, the $h_1(x)$ distribution adjusted the four data better than the $h_2(x)$ distribution (smaller values of AIC, AICC, BIC, MSE, MAD and MaxD); (2) The $h_1(x)$ and $h_2(x)$ distributions can be used to model symmetrical and asymmetrical unimodal data; (3) The $h_1(x)$ distribution it is most appropriate to adjust bimodal symmetrical and asymmetrical data, showing high flexibility which is not common in the literature on probability distributions, and this can be very important in practical applications. It is important to note that the distributions introduced in this paper can be used to create a formal statistical test to verify if the data has bimodality or not using a similar procedure introduced by Andrade and Rathie [2].

The paper is organized as follows. In Section 2, we introduce the $h_1(x)$ and $h_2(x)$ distributions and obtain the mathematical expressions for the moments of $h_1(x)$. In Section 3, we apply the new distributions in four real data sets. Finally, in Section 4, some final considerations of the results obtained in this paper are given.

We conclude this introduction section with some results which will be useful in the subsequent sections of this paper.

1.1   Azzalini's formula

Azzalini [3] obtained the following skew density function:

$$h(x) = 2f(x)G(w(x)) \quad (-\infty < x < \infty), \tag{1}$$

where $f(x)$ is a symmetric probability density function about the origin, $G(x)$ is the cumulative distribution function of a symmetric density function about the origin, and $w(x)$ is an odd function of x. In this paper, we take $w(x) = qx$, $q \in \mathbb{R}$.

1.2 Moments

It is easy to calculate the n-th moments of $h(x)$ given in (1) with $w(x) = qx$, $q \in \mathbb{R}$, which are

$$E(X^n) = 2 \int_0^\infty x^n f(x) dx, \tag{2}$$

when n is even, and

$$E(X^n) = 4 \int_0^\infty x^n f(x) G(qx) dx - 2 \int_0^\infty x^n f(x) dx, \tag{3}$$

when n is odd.

1.3 Rathie-Swamee generalized logistic distribution

In this section we define the symmetric generalized logistic density function and its cumulative distribution function studied recently by Rathie and Swamee [12]:

$$f(x) = \frac{[a + b(1+p)|x|^p] \exp\left[-x\left(a + b|x|^p\right)\right]}{\left\{\exp\left[-x\left(a + b|x|^p\right)\right] + 1\right\}^2}, \tag{4}$$

$$F(x) = \left\{\exp\left[-x\left(a + b|x|^p\right)\right] + 1\right\}^{-1}, \tag{5}$$

where $x \in \mathbb{R}, a \geq 0, b \geq 0, p > 0$ (with a and b are not zeros simultaneously), and $\mathbb{R}$ is the set of real numbers. For the values $a = 1.59413$, $b = 0.07443$ and $p = 1.939$, this distribution approximates very well the normal distribution with a maximum error of 4×10^{-4} at $x = 0$ for the density function and 7.757×10^{-5} at $x = 2.81$ for the distribution function. For approximations to Student-t distribution, see Rathie *et al.* [11]. The case $a = 0$ was studied, and applied to a civil engineering problem by Swamee and Rathie [14]. In the recent review article on univariate normal distribution, Rathie [13] point out that the generalized logistic distribution defined in (4) and (5) is invertible and that the approximation to the normal distribution is important for practical applications.

1.4   Eirado-Rathie distribution

In this section we define the symmetric distribution proposed by Eirado and Rathie [4] and its cumulative distribution function. The probability density function is given by

$$g(x) = \frac{\lambda |x|^{2\lambda-1}(\gamma|x| + m)}{(x^2 + 2\gamma|x| + m)^{\lambda+1}} \tag{6}$$

where $x \in \mathbb{R}$, and $\lambda > 0, \gamma, m \geq 0$ (γ and m are not zeros simultaneously). The cumulative distribution function is given by

$$G(x) = \frac{1}{2} + \frac{x|x|^{2\lambda-1}}{2(x^2 + 2\gamma|x| + m)^{\lambda}} \tag{7}$$

2   New skew distributions

2.1 $h_1(x)$ distribution

Using the Azzalini's formula 1, we can obtain the distribution $h_1(x)$, with $f(x)$ given by the equation (4) and $G(x)$ given by the equation (7). Then, the skew probability density function, $h_1(x)$ is given by

$$h_1(x) = \frac{[a + b(1+p)|x|^p]exp(-x(a + b|x|^p))}{[1 + exp(-x(a + b|x|^p))]^2}\Big(1 + \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + m)^{\lambda}}\Big), \tag{8}$$

where $x, q \in \mathbb{R}$, $q \neq 0$ and $a, b, \gamma, m \geq 0$, $p, \lambda > 0$ (a and b as well as γ and m are not zeros simultaneously). Plots for probability density function (8), for some values of q, a, b, p, λ, γ and m, showing different forms of the $h_1(x)$ distribution are illustrated in Figures 1 to 4. The density has symmetric, asymmetric to the left and asymmetric to the right behavior. It is interesting to note that, for values of the parameter a near to zero, the $h_1(x)$ distribution has a bimodal shape, which may be very important in practical applications.

2.1.1   Moments of h_1

The n-th moments of h_1 are given by,

$$E(X^n) = \int_{-\infty}^{\infty} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a + b|x|^p))}{[1 + exp(-x(a + b|x|^p))]^2}\Big(1 + \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^{\lambda}}\Big)dx$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a + b|x|^p))}{[1 + exp(-x(a + b|x|^p))]^2}dx$$
$$+ \int_{-\infty}^{\infty} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a + b|x|^p))}{[1 + exp(-x(a + b|x|^p))]^2}\frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^{\lambda}}dx,$$
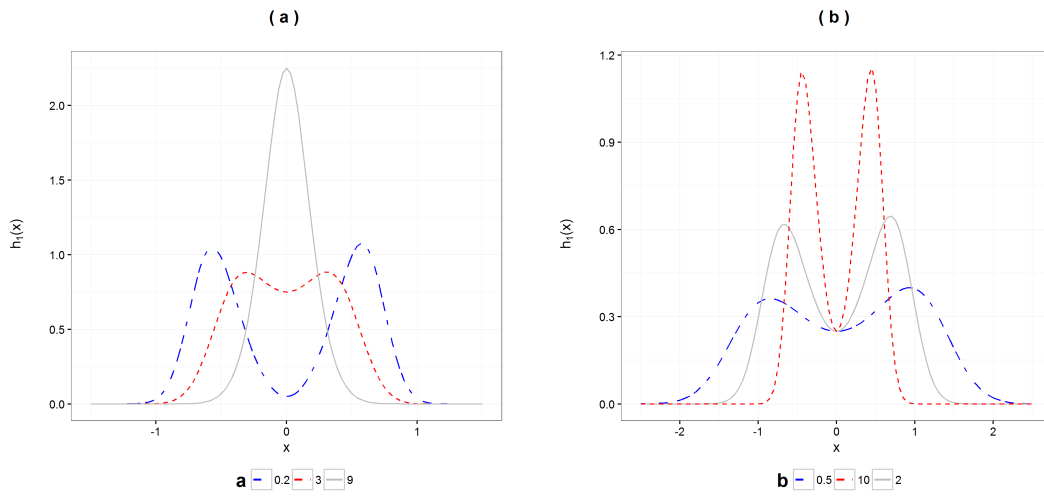
Figure 1: In (a), the parameter a assumes the values 0.2, 3 and 9, and $b = 6$, $p = q = m = 2$, $\lambda = 3$, $\gamma = 3$. Note that a changes bimodality of $h_1(x)$. In (b), the parameter b, takes the values 0.5, 2 and 10, and $a = \gamma = 1$, $p = q = m = 2$, $\lambda = 3$. The parameter b changes the shape of distribution.
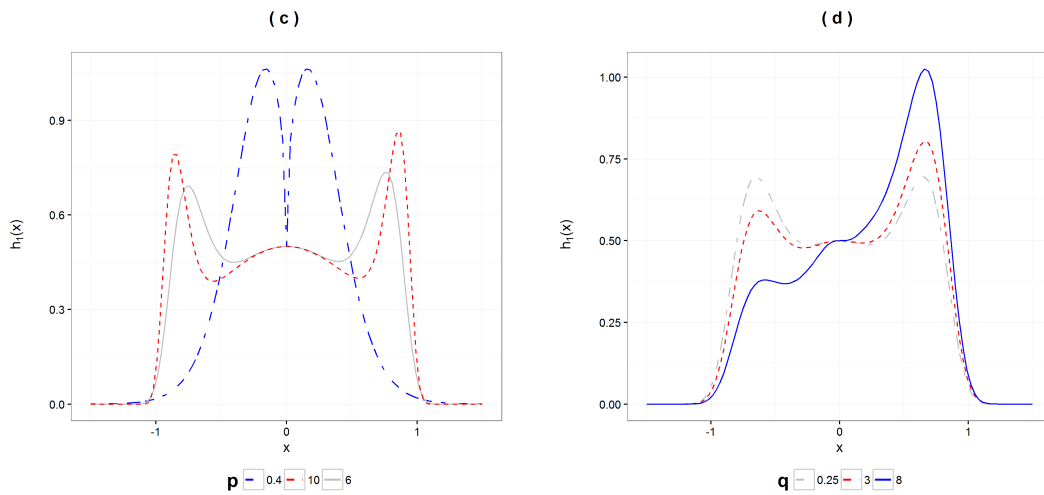

Figure 2: In (c), the parameter p assumes the values 0.4, 6 and 10, and $a = q = m = 2$, $b = 4$, $\lambda = 3$, $\gamma = 1$. See that, the parameter p affects the shape of distribution. In (d), the parameter q assume the values 0.25, 3 and 8 and $a = \lambda = m = 2$, $b = p = 4$, $\gamma = 1$. The parameter q affects the asymmetry of the distribution $h_1(x)$.
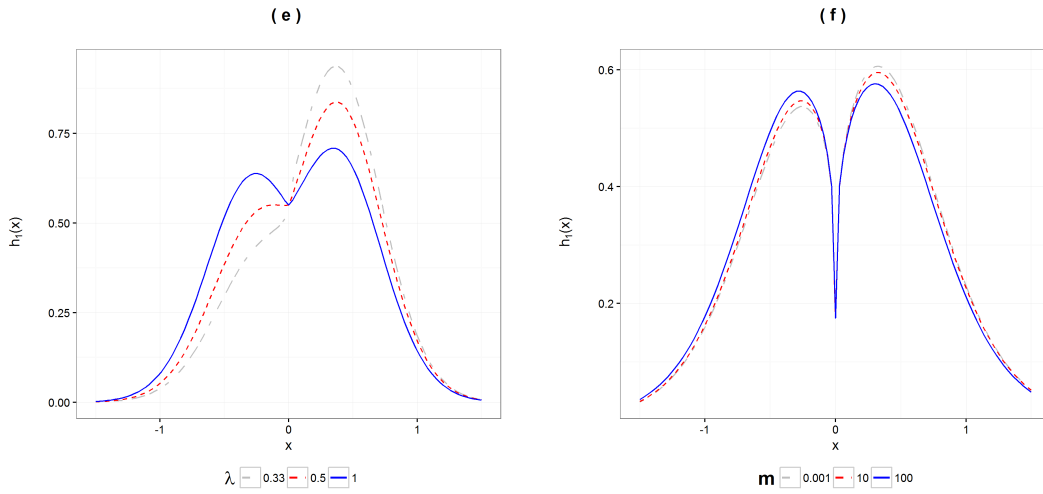
so we have the following integrals,

(e)

(f)

Figure 3: In (e), the parameter λ takes the values 0.33, 0.5 and 1, and $a =$ 2.2, $m = 1.2$, $q = 1$, $b = 1.75$, $p = 1.2$, $\gamma = 0.7$. One can see that the parameter λ affects the asymmetry of the distribution. In (f), the parameter m assume the values 0.001, 10 and 100 and $a = 0.7$, $\lambda = 1; q = 4$, $b = 2$, $p = 0.3$, $\gamma = 6$. See that, the parameter m is related to the asymmetry of the distribution $h_1(x)$.

(g)
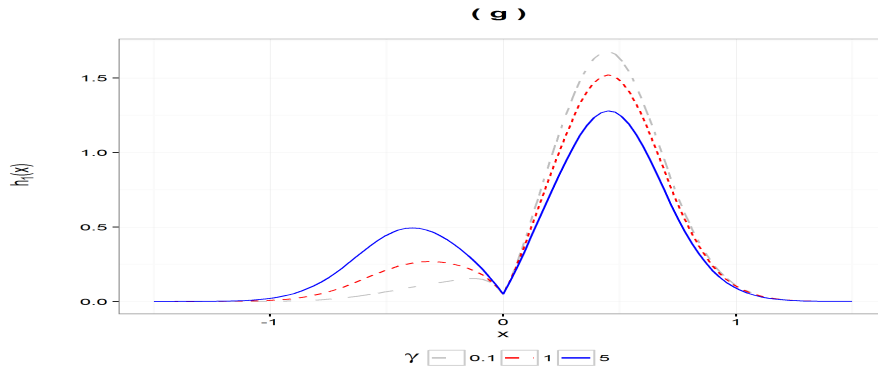
Figure 4: In (g), the parameter γ assumes the values 0.1, 1 and 5 and $a = 0.2$, $p = 1$, $m = 1$, $q = 6$, $b = 5, \lambda = 0.5$. One can see that the parameter γ affects the asymmetry of the distribution too.

$$I = \int_{-\infty}^{\infty} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} dx$$

and

$$II = \int_{-\infty}^{\infty} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^\lambda} dx$$

In the integral I we have the n-th moments of the Rathie-Swamee distribution (Rathie and Swamee, [12]) which are given by,

$$E(X^n) = 2\sum_{r=0}^{\infty}(-1)^r(1+r)[aI_{n,r} + b(1+p)I_{n+p,r}] \tag{9}$$

where,

$$I_{\alpha,r} = \int_0^\infty x^\alpha exp(-(1+r)x(\alpha+bx^p))dx. \tag{10}$$

The above integral, in terms of H-function, is given in Rathie and Swamee [12]. In integral II, we can get the following expression

$$
\begin{aligned}
II \; &= \; \int_{-\infty}^0 x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^\lambda} dx + \\
&\quad \int_0^\infty x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^\lambda} dx
\end{aligned}
$$

Partitioning the integral II in the integrals III and IV, we have

$$III \; = \; \int_0^\infty x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^\lambda} dx$$

To resolve the integral III, we use the following result,

$$(1+\alpha)^\beta = \sum_{r=0}^{\infty}(-\beta)_r \frac{(-\alpha)^r}{r!}.$$

So, the integral III can be written in the following form

$$
\begin{aligned}
III \; &= \; \int_0^\infty x^n[a + b(1+p)x^p]exp(-x(a+bx^p)) \sum_{r=0}^{\infty}(2)_r \frac{(-1)^r exp(-xr(a+bx^p))}{r!} \\
&\quad \frac{(qx)^{2\lambda}}{((qx)^2 + 2\gamma qx + w)^\lambda} dx \\
&= \; \sum_{r=0}^{\infty} \frac{(2)_r}{r!}(-1)^r q^{2\lambda} \int_0^\infty x^{n+2\lambda} \frac{[a + b(1+p)x^p]exp(-(r+1)x(a+bx^p))}{((qx)^2 + 2\gamma qx + w)^\lambda} dx.
\end{aligned}
$$

The integral IV is

$$
\begin{aligned}
IV &= \int_{-\infty}^{0} x^n \frac{[a + b(1+p)|x|^p]exp(-x(a+b|x|^p))}{[1 + exp(-x(a+b|x|^p))]^2} \frac{qx|qx|^{2\lambda-1}}{((qx)^2 + 2\gamma|qx| + w)^\lambda} dx \\
&= \int_{0}^{\infty} (-x)^n \frac{[a + b(1+p)x^p]exp(x(a+bx^p))}{[1 + exp(x(a+bx^p))]^2} \frac{q(-x)(qx)^{2\lambda-1}}{((qx)^2 + 2\gamma qx + w)^\lambda} dx \\
&= (-1)^{n+1} q^{2\lambda} \int_{0}^{\infty} \frac{[a + b(1+p)x^p]exp(-x(a+bx^p))}{[1 + exp(-x(a+bx^p))]^2} \frac{x^{2\lambda+n}}{((qx)^2 + 2\gamma qx + w)^\lambda} dx \\
&= (-1)^{n+1} q^{2\lambda} \int_{0}^{\infty} [a + b(1+p)x^p]exp(-x(a+bx^p)) \sum_{r=0}^{\infty} (2)_r \frac{(-1)^r exp(-rx(a+bx^p)))}{r!} \\
&\quad \frac{x^{2\lambda+n}}{((qx)^2 + 2\gamma qx + w)^\lambda} dx \\
&= \sum_{r=0}^{\infty} \frac{(2)_r}{r!} (-1)^{n+r+1} q^{2\lambda} \int_{0}^{\infty} [a + b(1+p)x^p]exp(-x(a+bx^p))(exp(-rx(a+bx^p))) \\
&\quad \frac{x^{2\lambda+n}}{((qx)^2 + 2\gamma qx + w)^\lambda} dx \\
&= \sum_{r=0}^{\infty} \frac{(2)_r}{r!} (-1)^{n+r+1} q^{2\lambda} \int_{0}^{\infty} x^{n+2\lambda} \frac{[a + b(1+p)x^p]exp(-(1+r)x(a+bx^p))}{((qx)^2 + 2\gamma qx + w)^\lambda} dx
\end{aligned}
$$

Let

$$
Y_{\alpha,r} = \int_{0}^{\infty} x^\alpha \frac{exp(-(r+1)x(a+bx^p))}{((qx)^2 + 2\gamma qx + w)^\lambda} dx, \tag{11}
$$

an integral to be calculated numerically for given values of parameters. We get, for even n,

$$
\begin{aligned}
E(X^n) &= 2\sum_{r=0}^{\infty} (-1)^r (1+r)[aI_{n,r} + b(1+p)I_{n+p,r}] + \sum_{r=0}^{\infty} \frac{(2)_r}{r!} (-1)^{n+r+1} q^{2\lambda} \times \\
&\quad [aY_{n+2\lambda,r} + b(1+p)Y_{n+p+2\lambda,r}] + \sum_{r=0}^{\infty} \frac{(2)_r}{r!} (-1)^r q^{2\lambda}[aY_{n+2\lambda,r} + b(1+p)Y_{n+p+2\lambda,r}] \\
E(X^n) &= 2\sum_{r=0}^{\infty} (-1)^r (1+r)[aI_{n,r} + b(1+p)I_{n+p,r}] + \\
&\quad ((-1)^{n+1} + 1)\sum_{r=0}^{\infty} \frac{(2)_r}{r!} (-1)^r q^{2\lambda}[aY_{n+2\lambda,r} + b(1+p)Y_{n+p+2\lambda,r}]
\end{aligned}
$$

and using the results of Rathie and Swamee [12], we have

$$E(X^n) \quad = \quad 2\sum_{r=0}^{\infty}(-1)^r(1+r)[aI_{n,r} + b(1+p)I_{n+p,r}]. \tag{12}$$

For odd n, the moments are given by

$$E(X^n) = 2\sum_{r=0}^{\infty}\frac{(2)_r}{r!}(-1)^r q^{2\lambda}[aY_{n+2\lambda,r} + b(1+p)Y_{n+p+2\lambda,r}]$$

And, finally, we have the following expressions for the n-th moments of $h_1(x)$,

$$E(X^n) = \begin{cases} 2\sum_{r=0}^{\infty}(-1)^r(1+r)[aI_{n,r} + b(1+p)I_{n+p,r}], & \text{for even } n, \\ 2\sum_{r=0}^{\infty}(-1)^r\dfrac{(2)_r}{r!}q^{2\lambda}[aY_{n+2\lambda,r} + b(1+p)Y_{n+p+2\lambda,r}], & \text{for odd } n. \end{cases} \tag{13}$$

It is important to note that the even moments are given in terms of H function and the odd moments are to be calculated numerically. For more details about H function the reader can consult the following references: Luke [6] and Mathai *et al.* [7].

2.2 $h_2(x)$ distribution

Using (1), we get the $h_2(x)$ distribution, by using $g(x)$ as the probability density function Eirado-Rathie (6) and $H(x)$ as the cumulative distribution function of Generalized logistic (5). Thus, the following probability density function is obtained,

$$h_2(x) = \frac{2\lambda|x|^{2\lambda-1}(\gamma|x| + m)}{(x^2 + 2\gamma|x| + m)^{\lambda+1}[1 + exp(-qx(a + b|qx|^p))]} \tag{14}$$

where x and $q \in \mathbb{R}, q \neq 0$, a, b, $\gamma, m \geq 0$ and $p, \lambda > 0$ (a and b as well as γ and m are not zeros simultaneously). Plots for probability density function (14), for some values of q, a, b, p, λ, γ and m, to show different forms of the $h_2(x)$ distribution are given in Figures 5 to 7. The density has symmetric, asymmetric to the left and asymmetric to the right behaviour. It is interesting to note that, for some values of parameters, the $h_2(x)$ distribution has a bimodal and trimodal shape, which may be very important in practical applications.
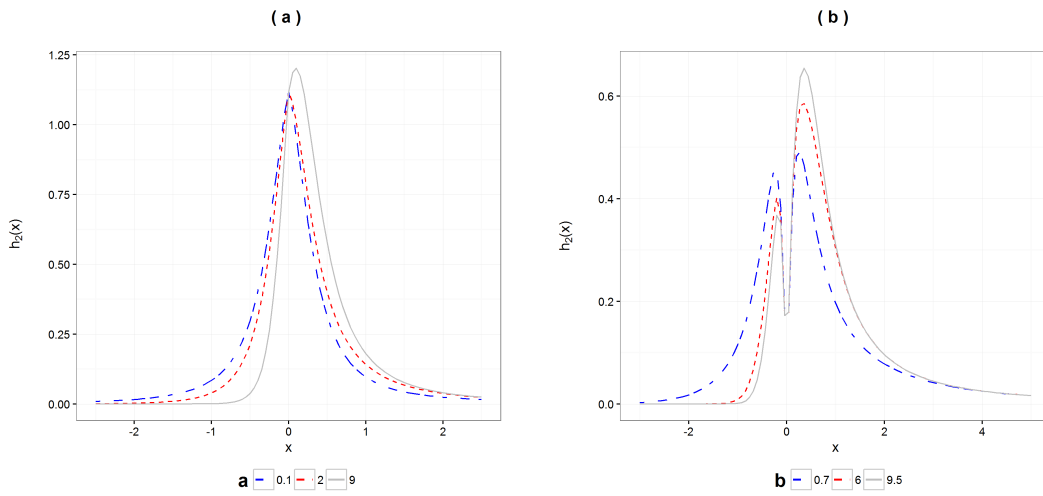
Figure 5: In (a), the parameter a takes the values $0.1, 2, 9$, and $b = q = 0.2$, $p = 1$, $m = 0.6$, $\lambda = 0.5, \gamma = 0.1$. It may be noted that, the parameter a changes the asymmetry. In (b), the parameter b assumes the values $0.7, 6, 9.5$, and $a = \gamma = q = 0.2$, $p = 0.6$, $m = 0.7, \lambda = 1$. We note that the parameter b also changes the asymmetry and the appearance of modes of $h_2(x)$.
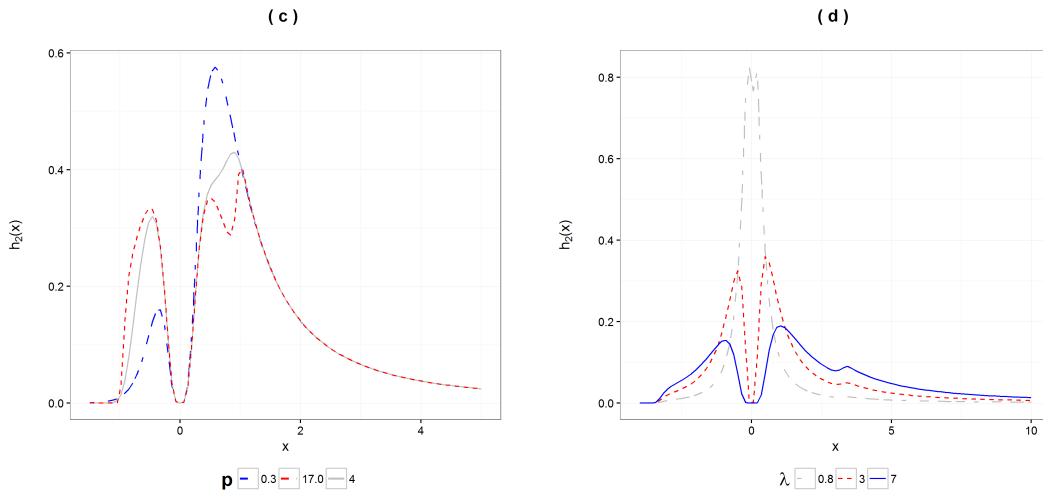

Figure 6: In (c), the parameter p takes the values $0.3, 4, 17$, and $a = \gamma = q = 0.1$, $b = 3.6$, $m = 1, \lambda = 3$. Note that, the parameter p influences the asymmetry. In (d), the parameter γ takes the values $0.3, 0.9, 2.4$, and $a = m = q = 0.1$, $p = 0.3$, $b = 2, \lambda = 1$. Note that the parameter γ changes the shape of distribution $h_2(x)$.

3   Applications to Real Data

In this section, we apply the distributions introduced in this paper to four real data sets. The first data set is related to the total spending on public
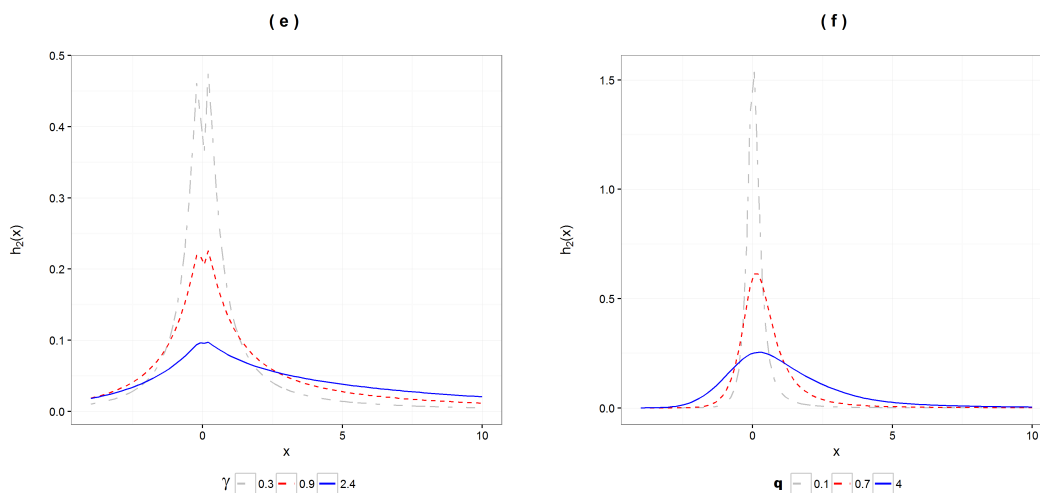
Figure 7: In (e), the parameter q takes the values $0.1, 0.7, 4$, and $a = 2.4$, $b = 0.6$, $p = 2$, $m = \lambda = 0.5, \gamma = 0.01$. Note that the parameter q changes the shape of distribution. In (f), parameter λ assumes the values $0.8, 3, 7$, and $a = 0.7$, $b = 1.8$, $p = 20, m = 0.3, \gamma = q = 0.01$. One can note that the parameter λ influences the distance of the modes.

education (% of GDP—Gross Domestic Product) in various countries in 2003, which is unimodal and asymmetrical. The second data set relates to the total expenditure, in 2009, on health (% of GDP—Gross Domestic Product) in various countries, which has a unimodal and asymmetric behavior. The third data set is related to the waiting time between eruptions of the Old Faithful Geyser in the Yellow Stone National Park, Wyoming, USA, which is clearly bimodal and asymmetrical. And finally, the fourth data set is related to pH concentration in coal mine abandoned in Pennsylvania, the data is reported by Growitz *et al.* [5].

The performance of the models was then compared by using the Akaike criterion (AIC), Bayesian criterion (BIC), Modified Akaike criterion (AICC) and Komogorov-Sminorv test (KS-Test). The information criterion AIC, BIC and AICC are given by

$$
\begin{aligned}
AIC &= -2\log(f(\mathbf{x}|\boldsymbol{\theta})) + 2p; \\
BIC &= -2\log(f(\mathbf{x}|\boldsymbol{\theta})) + p\log(n); \\
AICC &= -2\log(f(\mathbf{x}|\boldsymbol{\theta})) + 2\frac{p(p+1)}{n-p-1},
\end{aligned}
\tag{15}
$$

where $\log(f(\mathbf{x}|\boldsymbol{\theta}))$ is the log-likelihood function, p is the number of parameters of models and n is the sample size. The models that have lowest AIC, BIC and AICC values are better.

The accuracy of the models was then compared by using Mean Square Error (MSE), Mean Deviation Absolute (MDA) and Max Deviation (MaxD). The MSE, MDA and MaxD are given by

$$
\begin{aligned}
MSE &= \frac{\sum_{i=1}^{n} (F_e(x_i) - \hat{F}(x_i))^2}{n} \\
MAD &= \frac{\sum_{i=1}^{n} |F_e(x_i) - \hat{F}(x_i)|}{n} \\
MaxD &= \max(|F_e(x_i) - \hat{F}(x_i)|), \quad i = 1, \ldots, n,
\end{aligned}
\tag{16}
$$

where $F_e(x_i)$ is the empirical cumulative distribution and $\hat{F}(x_i)$ is the fitted cumulative distribution of the data. The models that have minimum values of MSE, MAD and MaxD (close to zero) are better.

To adjust this data set, we modify the models by introducing a location parameter μ and a scale parameter σ by changing x to $(x - \mu)/\sigma$ everywhere in the density function divided by σ. The software R was used to calculate the estimates of the parameters through maximum likelihood method and the R function *constrOptim* [8] was used to maximize the log-likelihood function. The reason for using the R function *constrOptim* is to guarantee that the estimated parameters are consistent within their respective parametric space.

3.1   Application 1: Expenditure on Education

We use the data of total spending on public education (% of GDP- Gross Domestic Product) in various countries in 2003. These data were obtained from the site [15]. Expenditure on public education includes the current and capital spending by private and government agencies on educational institutions (both public and private), educational administration and subsidies to private (student/family) entities.

The maximum likelihood estimates for the parameters of the models are given by:

- $h_1(x)$ distribution: $\hat{a} = 1.755050$, $\hat{b} = 4.013829 \times 10^{-02}$, $\hat{p} = 2.006134 \times 10^{-04}$, $\hat{q} = 3.247829$, $\hat{\lambda} = 18.48830$, $\hat{\gamma} = 2.792748 \times 10^{-06}$, $\hat{m} = 0.3820827$, $\hat{\mu} = 3.908747$ and $\hat{\sigma} = 2.112642$;

- $h_2(x)$ distribution: $\hat{a} = 3.220658 \times 10^{-06}$, $\hat{b} = 1.113516 \times 10^{-04}$, $\hat{p} = 103.8176$, $\hat{q} = 0.2322895$, $\hat{\lambda} = 1.201950$, $\hat{\gamma} = 1.300959 \times 10^{-05}$, $\hat{m} = 2.134683$, $\hat{\mu} = 4.723628$ and $\hat{\sigma} = 0.7138086$.

The Figure 8 illustrates the fit of the distributions introduced in this paper. The Figure 9 illustrates the $pp - plot$ of all distributions. The performance of the fitted distributions are given in Table 1.
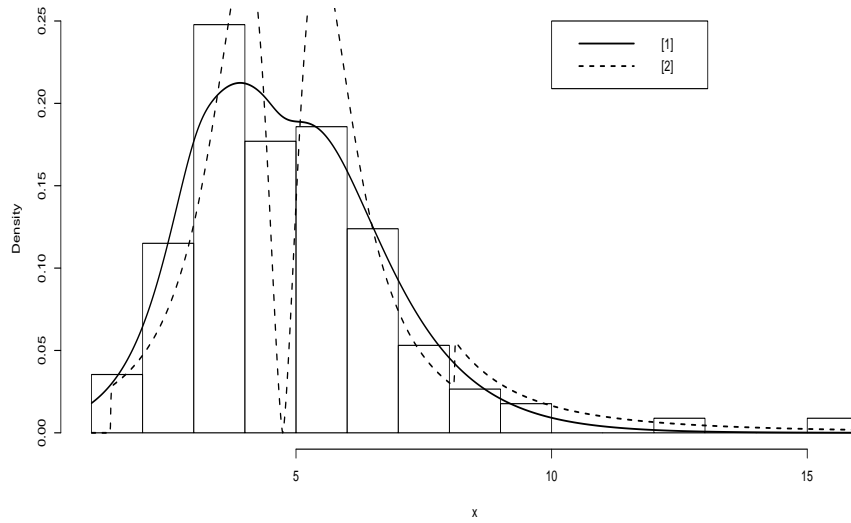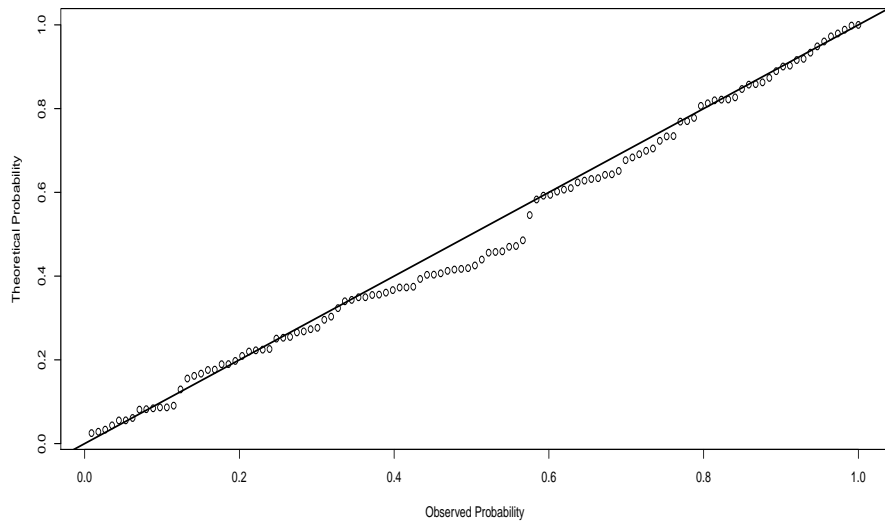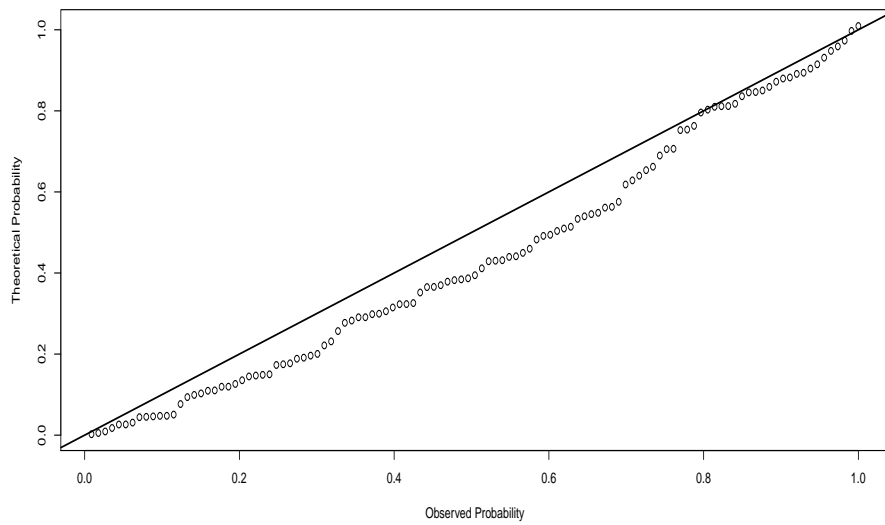
(a)
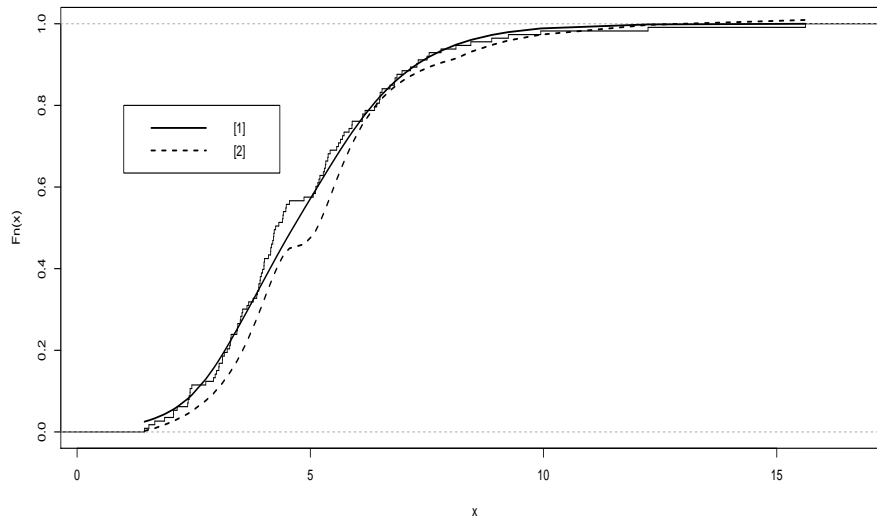
(b)

Figure 8: Education data - Fitted distributions. [1] $h_1(x)$ distribution; [2] $h_2(x)$ distribution. (**a**) Probability density function; (**b**) Cumulative distribution.

(a)

(b)

Figure 9: Education data - PP-Plots. (**a**) $h_1(x)$ distribution; (**b**) $h_2(x)$ distribution.

Table 1: Performance and accuracy of the distributions.

Model	AIC	BIC	AICC	KS-Test (p-Value)	MSE	MAD	MaxD
$h_1(x)$	486.135	510.6815	469.8826	0.7681	0.0009	0.0207	0.0856
$h_2(x)$	472.5714	497.1179	456.319	0.351	0.0063	0.0716	0.1292

3.2 Application 2: Expenditure on Health

We use the data of total expenditure, in 2009, on health (% of GDP—Gross Domestic Product) in various countries. These data are obtained from the site [16]. Total health expenditure is the sum of expenses with public and private health. It covers the provision of health services (preventive and curative), family planning activities, nutrition activities and emergency aid designated for health but does not include water supply and sanitation.

The maximum likelihood estimates for the parameters of the models are given by:

- $h_1(x)$: $\hat{a} = 2.214195 \times 10^{-02}$, $\hat{b} = 2.694633$, $\hat{p} = 1.842478 \times 10^{-06}$, $\hat{q} = 3.663260$, $\hat{\lambda} = 42.63227$, $\hat{\gamma} = 3.079846 \times 10^{-07}$, $\hat{m} = 0.2124404$, $\hat{\mu} = 6.027183$ and $\hat{\sigma} = 4.288508$;

- $h_2(x)$: $\hat{a} = 0.5722678$, $\hat{b} = 864.8491$, $\hat{p} = 3.461019$, $\hat{q} = 0.1762805$, $\hat{\lambda} = 0.4251121$, $\hat{\gamma} = 6.113476 \times 10^{-03}$, $\hat{m} = 0.8612122$, $\hat{\mu} = 6.077079$ and $\hat{\sigma} = 3.249853$.

The Figure 10 illustrates the fit of the distributions introduced in this paper. The Figure 11 illustrates the $pp-plot$ of both distributions. The performance of the fitted distributions are illustrated in Table 2.

Table 2: Performance and accuracy of the distributions.

Model	AIC	BIC	AICC	KS-Test (p-Value)	MSE	MAD	MaxD
$h_1(x)$	1040.04	1016.26	1050.52	0.6887	0.0005	0.0189	0.0648
$h_2(x)$	1073.552	1104.136	1056.405	0.3876	0.0021	0.0412	0.0842

3.3 Application 3: Waiting Time between Eruptions of Old Faithful Geyser

This application shows the versatility of the $h_1(x)$ and the $h_2(x)$ distributions. Using data available in the free statistical software R we see the bimodal shape of the distribution. Among the variables available, the waiting time between
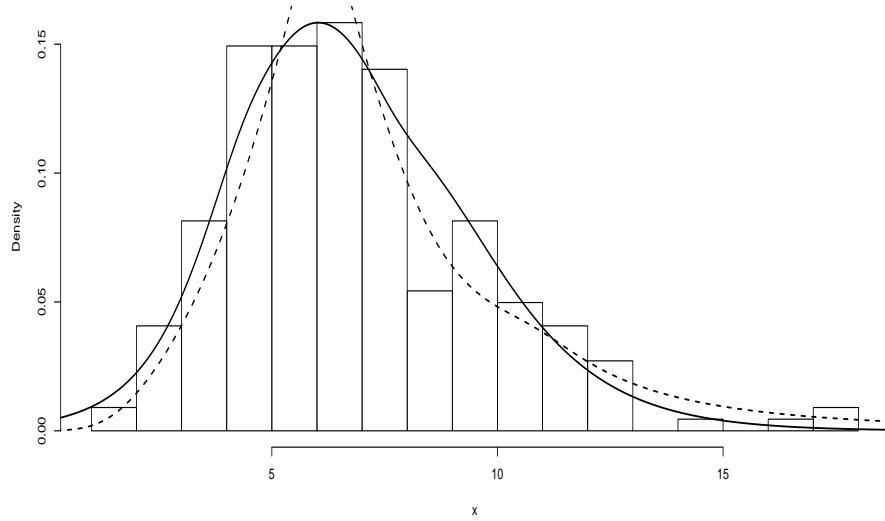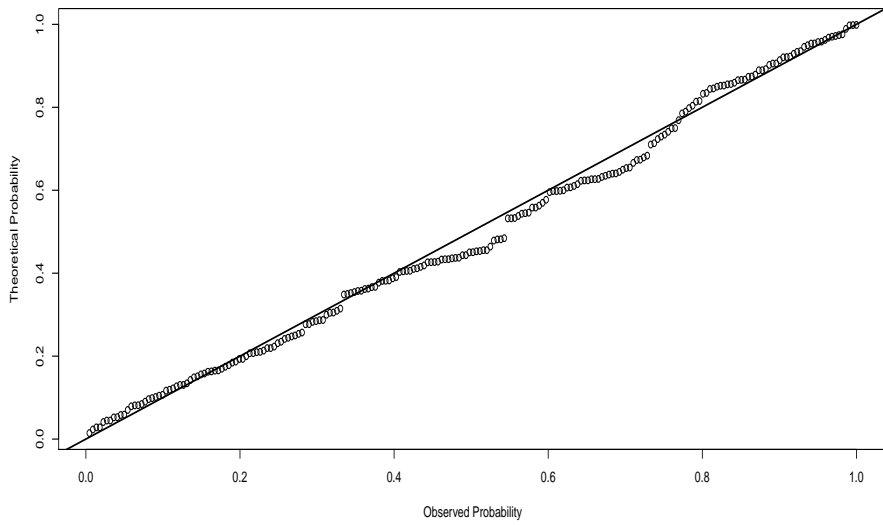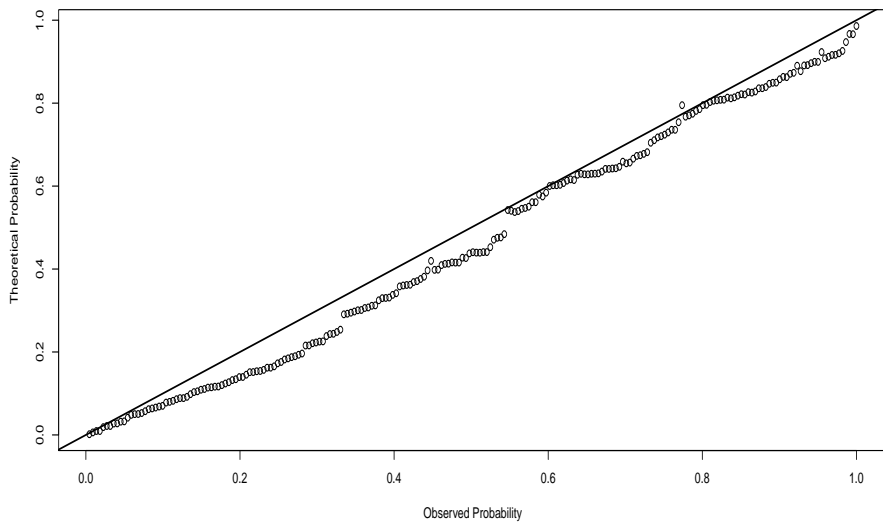
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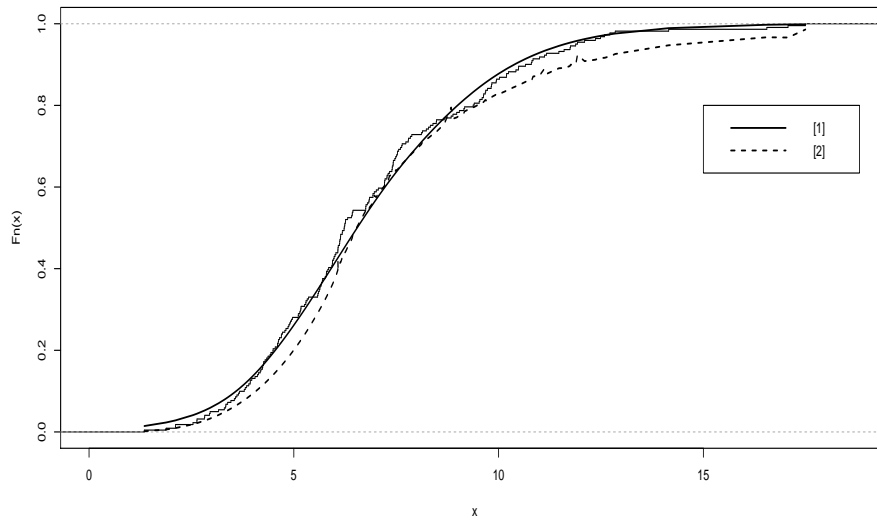
(b)

Figure 10: Health data - Fitted distributions. [1] $h_1(x)$ distribution; [2] $h_2(x)$ distribution. (**a**) Probability density function; (**b**) Cumulative distribution.

(a)

(b)

Figure 11: Health data - PP-Plots. (**a**) $h_1(x)$ distribution; (**b**) $h_2(x)$ distribution.

eruptions of Old Faithful Geyser in Yellow Stone National Park, Wyoming, USA was used. The data has 272 observations given in minutes.

The maximum likelihood estimates for the parameters of the models are given by:

- $h_1(x)$ distribution: $\hat{a} = 0.4008738$, $\hat{b} = 1.5289975$, $\hat{p} = 1.5447758$, $\hat{q} = 1.3598397$, $\hat{\lambda} = 0.5916388$, $\hat{\gamma} = 3.0383708$, $\hat{m} = 1.4931424$, $\hat{\mu} = 66.8590271$ and $\hat{\sigma} = 16.8678110$;

- $h_2(x)$ distribution: $\hat{a} = 0.4284459$, $\hat{b} = 9.590022 \times 10^{-06}$, $\hat{p} = 16.02935$, $\hat{q} = 1.306598$, $\hat{\lambda} = 1.408497$, $\hat{\gamma} = 5.144536 \times 10^{-11}$, $\hat{m} = 0.3180452$, $\hat{\mu} = 67.45358$ and $\hat{\sigma} = 16.79740$.

The Figure 12 illustrates the fit of the distributions introduced in this paper. The Figure 13 illustrates the $pp - plot$ of our distributions. The performance of the fitted distributions are given in Table 3.

Table 3: Performance and accuracy of the distributions.

| Model	AIC	BIC	AICC	KS-Test (p-Value)	MSE	MAD	MaxD
$h_1(x)$	2085.139	2117.591	2067.826	0.7344	0.0004	0.0173	0.0381
$h_2(x)$	2218.834	2251.286	2201.521	1.967×10^{-05}	0.0081	0.0756	0.1728

3.4 Application 4: pH concentration

Values for pH, alkalinity, acidity, sulphate and metals in groundwater concentration are available from drainage samples of coal mine abandoned in Pennsylvania. Data are from 252 discharge mines, reported by Growitz *et al.* [5].

The maximum likelihood estimates for the parameters of the models are given by:

- $h_1(x)$ distribution: $\hat{a} = 2.02048840$, $\hat{b} = 7.87540739$, $\hat{p} = 2.48477812$, $\hat{q} = 0.02650005$, $\hat{\lambda} = 0.77128388$, $\hat{\gamma} = 1.39435821$, $\hat{m} = 0.39311383$, $\hat{\mu} = 4.91580660$ and $\hat{\sigma} = 2.73918977$;

- $h_2(x)$ distribution: $\hat{a} = 8.759654 \times 10^{-08}$, $\hat{b} = 5.629886 \times 10^{-07}$, $\hat{p} = 17.36824$, $\hat{q} = 1.449127$, $\hat{\lambda} = 1.003071$, $\hat{\gamma} = 1.688870 \times 10^{-08}$, $\hat{m} = 0.3743292$, $\hat{\mu} = 4.949162$ and $\hat{\sigma} = 1.524822$.

The Figure 14 illustrates the fit of the distributions introduced in this paper. The Figure 15 illustrates the $pp - plot$ of our distributions. The performance of the fitted distributions are given in Table 4.
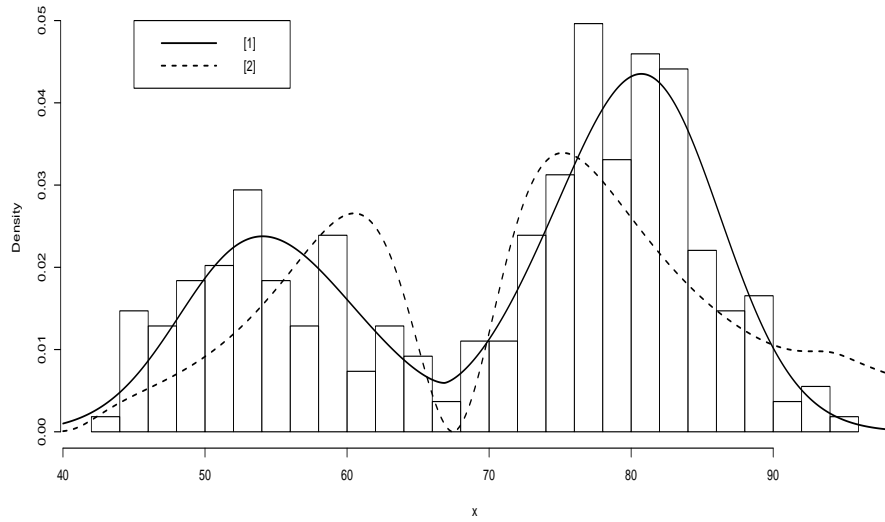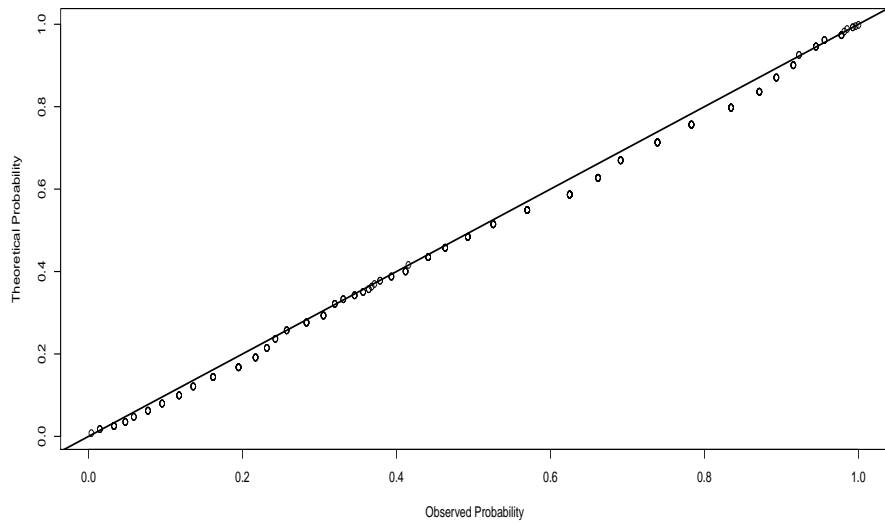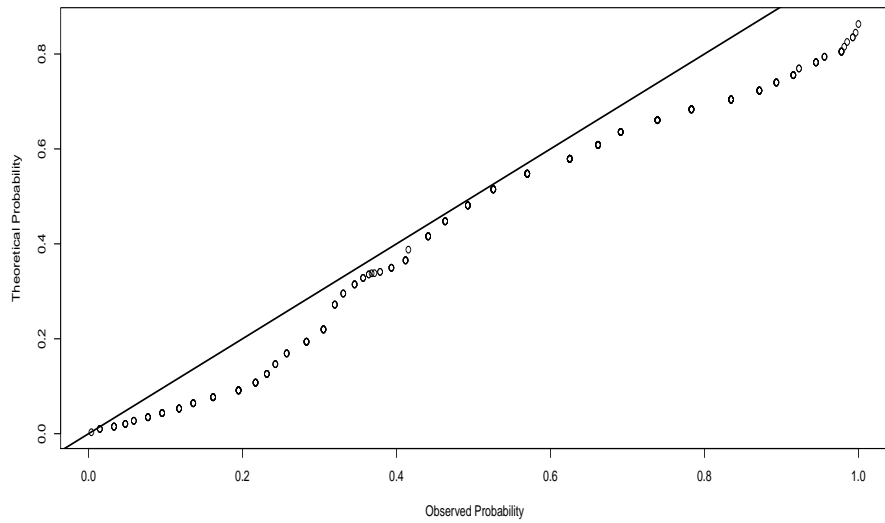
(a)
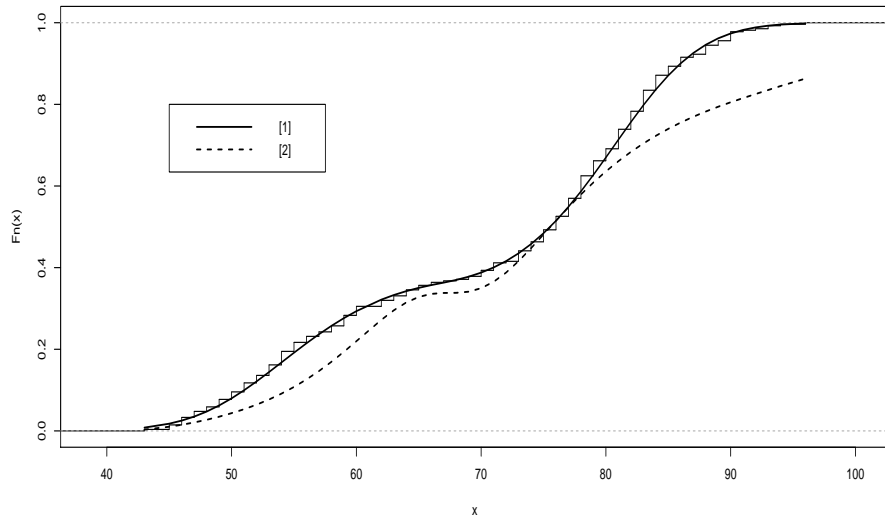
(b)

Figure 12: Faithful data - Fitted distributions. [1] $h_1(x)$ distribution; [2] $h_2(x)$ distribution. (**a**) Probability density function; (**b**) Cumulative distribution.

(a)

(b)

Figure 13: Faithful data - PP-Plots. (**a**) $h_1(x)$ distribution; (**b**) $h_2(x)$ distribution.

(a)

(b)

Figure 14: pH concentration data - Fitted distributions. [1] $h_1(x)$ distribution; [2] $h_2(x)$ distribution. (**a**) Probability density function; (**b**) Cumulative distribution.
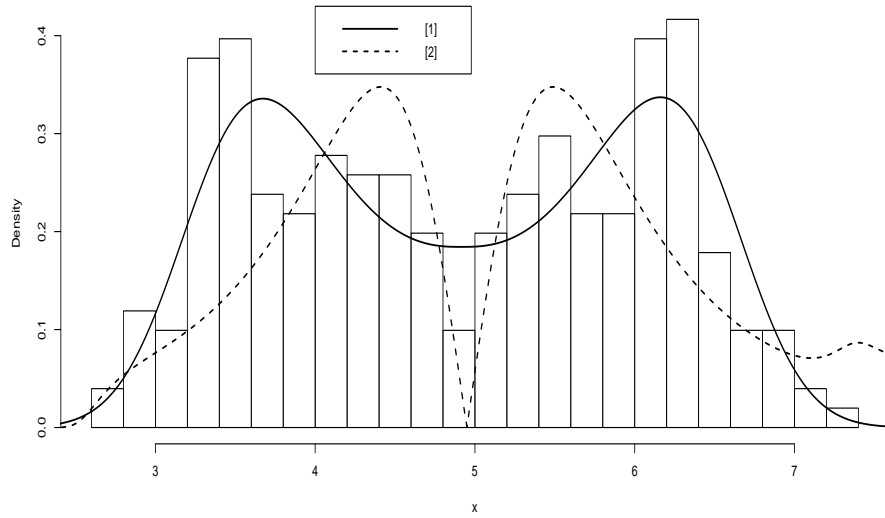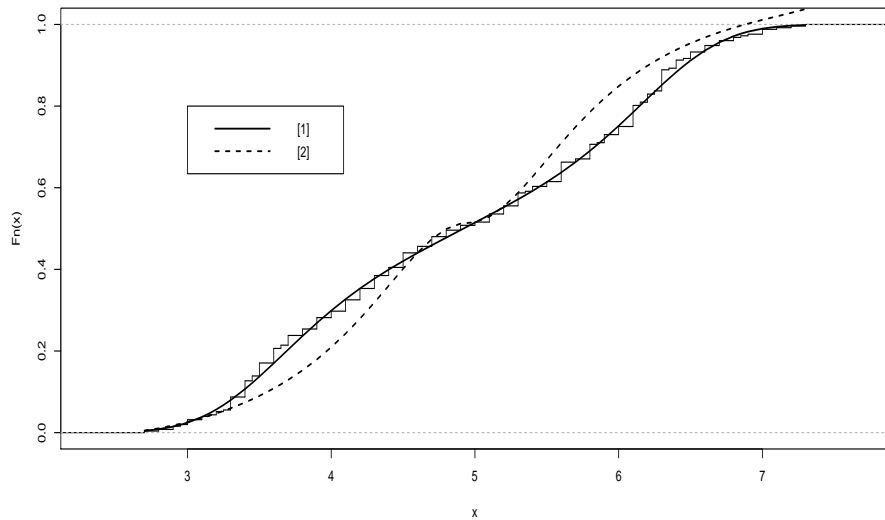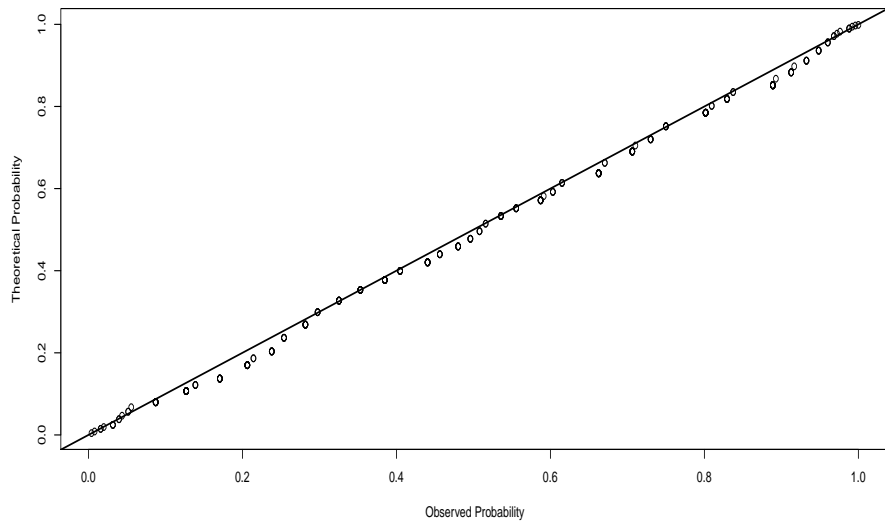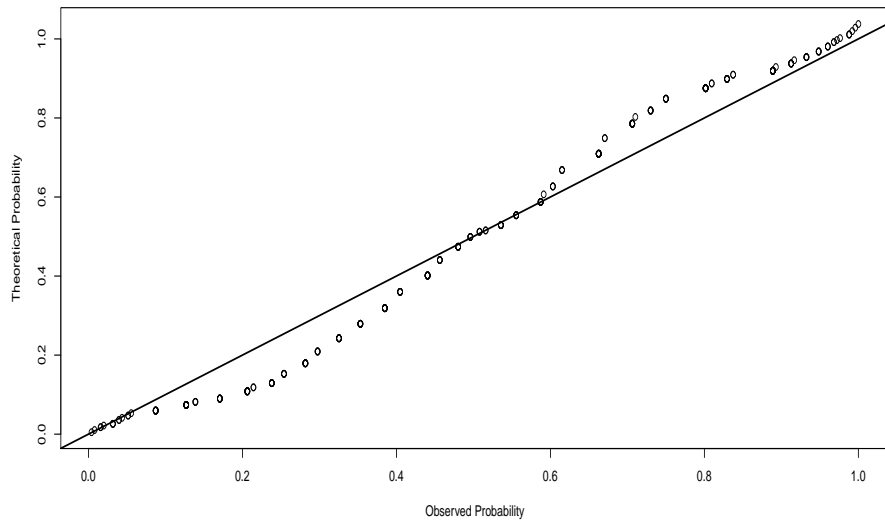
(a)

(b)

Figure 15: pH concentration data - PP-Plots. (**a**) $h_1(x)$ distribution; (**b**) $h_2(x)$ distribution.

Table 4: Performance and accuracy of the distributions.

Model	AIC	BIC	AICC	KS-Test (*p*-Value)	MSE	MAD	MaxD
$h_1(x)$	744.6913	776.4561	727.4351	0.8316	0.0003	0.0152	0.0373
$h_2(x)$	857.387	889.1519	840.1309	0.09438	0.0096	0.0912	0.1454

Observing the results in Tables 1 and 2 we can see that, looking the p-value of the KS test, the distributions can be used to model the data, fixing the significance level in 5%. From Table 1, according to the accuracy, the $h_1(x)$ distribution gave better accuracy compared to $h_2(x)$ distribution (Smaller values of MSE, MAD and MaxD), however, $h_2(x)$ distribution indicated better performance than $h_1(x)$ distribution (Smaller values of AIC, BIC and AICC).

Observing the results in Tables 2, 3 and 4, the $h_1(x)$ distribution presented better results when compared to the $h_2(x)$ distribution (Smaller values of AIC, BIC, AICC, MSE, MAD and MaxD).

4 Conclusions

In this paper, we proposed two new skew probability density functions using the Azzalini's formula $2f(x)G(cx)$, where f is a symmetric density about zero, and G is a distribution function of a symmetric density about zero. The expressions for f and G are taken Rathie-Swamee generalized logistic and Eirado-Rathie distributions. We derived expressions for the n-th moments in terms of the H and Meijer G functions [6, 7].

We apply new distributions to four data sets. Two applications for unimodal data are provided for total expenditure on education in various countries in 2003 and expenditure on health in various countries in 2009. Two applications of bimodal data are given for the waiting time between eruptions of the Old Faithful Geyser and pH concentration. We conclude that:

1. In general, the $h_1(x)$ distribution adjusted the four data sets better than the $h_2(x)$ distribution (smaller values of AIC, AICC, BIC, MSE, MAD and MaxD);

2. The $h_1(x)$ and $h_2(x)$ distributions can be used to model symmetrical and asymmetrical unimodal data;

3. The $h_1(x)$ distribution it is most appropriate to adjust bimodal symmetrical and asymmetrical data, showing high flexibility which is not common in the literature on probability distributions, and this can be very important in practical applications;

4. The distributions introduced in this paper can be used to create a formal statistical test to verify if the data has bimodality or not using a similar procedure introduced by Andrade and Rathie [2].

Thus, the proposed distributions in this paper are flexible to adjust symmetric and asymmetric data, with unimodal and bimodal behavior.

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