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On the Hamiltonian Bigraphs

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Abstract

In this paper we study and discuss simple Hamiltonian biographs and construct a relation of partition of Hamiltonian biographs into independent cycles under certain conditions, atangable results of partition of $K_{n,n}$ into two independent cycles is achieved .

Basic definitions

In this sequel we will introduce basic definitions and concepts of graph theory

Which will be used in this paper, all definitions in this paper from (Harary, 1994)

Definition1.1: A graph G=(V,E)consist of a finite nonempty set of vertices, together with a prescribed set of edges.

Definition 1.2: The number of vertices in G is called order of G. denoted by |V(G)|.

Definition 1.3: If a graph G is undirected without loops and parallel edges it called simple graph.

Definition1.4: A wallk with distinct edges and distinct vertices is called a path.

Definition 1.5: A cycles is closed path

Definition 1.6: The length of a cycle equal number of edges in it.

Definition 1.7: A bigraph G is a graph with two disjoint sets of vertices A and B, with |V(A)|=m|V(B)|=n .

Definition 1.8: A graph G is connected if there exist at least one path between any two vertices in G .

Definition 1.9: A connected graph G is Hamiltonian graph if there is a cycle covers all vertices in G.

Definition 1.10: The distance d(u,v) between two vertices u and v in G is the length of the shortest path between u and v in G.

Definition 1.11: let A be a subgraph of G and let v∈ V(G). Then the distance d(A,v)=d(v) is defined as follows:

$$d(v) = d(A,v) = \begin{cases} \min(d(v,u) \; if \; v \notin V(A), u \in V(A) \\ 0 \qquad\qquad\qquad\qquad if \; v \in V(A) \end{cases}$$

Definition 1.12:A biograph with partition (A,B) is called balance if |A|=|B| i.e $K_{n,n}$

Partition of Hamiltonian Biagraph into independent cycles

In this sequel we will discuss finite simple graph, (Bondy & Chvatal, 1976) illustrate that if G is a bipartite graph $K_{n,n}$ with bipartion (A,B) and for any x ∈ A , y∈ B and d(x) + d(y) ≥ n+1 Then G is Hamiltonian graph

Our main target is to prove the following result;

Theorem 4.1:

If $K_{n,n}$ bipartion into A and B such that for any x ∈ A , y ∈ B and d(x) + d(y) ≥ n+2, then for any (n_1,n_2, n= n_1+n_2)

G contains two independent cycles of length $2n_1$,$2n_2$.

Note: The conditions on theorem 4.1 gives that G is Hamiltoians (Amar, 1986)

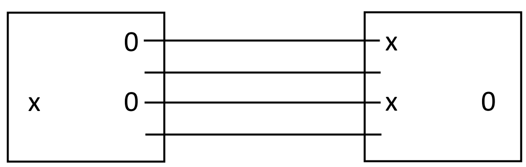
Notation:

If *F* and *H* are disjoint subgraphs of *G* and if $u_1,...,u_p$ are vertices of *G*, not in *H*, then $H+(u_1,...,u_P)$ is the subgraph of *G* with vertices $u_1,...,u_P$ and the vertices of *H*.

If v_l, ..., v_k are vertices of *H*, *H*-(v_l,...,v_k) is the subgraph of *G* with vertices in *H* except v_l, ..., v_k. If *C* is a path or a cycle of *G*, then for any arbitrary orientation; if *u* is a vertex of *C*, u^+ (resp. u^-) is the successor (resp. the predecessor) of u on the path or the cycle for the given orientation.

Remark:

If *n* is odd, the result of the theorem is the best one as we can see with the following example: *n*=2*p*+1. There is no partition into two cycles of lengths 2*p* and 2*p*+2 (see e.g. sceam.1).



sceam.1

It is clear that every vertex in *G* in sceam.1 of degree 4, so *d*(*x*)+ *d*(*y*)=8 and $|V(G)|=14$ but *G* does not contain two independent cycles of lengths 6, 8.

The proof of Theorem is based on many elementary lemmas that we wil give first.

Note: the proof of the following lemmas in thesis"The Cycles of simple Graph"ALrawajfeh,alaa.2012

Elementary Lemmas1.1:

Let G be a balanced bipartite graph with bipartition (A, B), such that $|A| = |B| = n$.

Lemma1.1.:

If for any $x \in A, y \in B, d(x,G) + d(y,G) \geq n+1$, then G is Hamiltionian.

Lemma2.1:

If G contains a Hamiltionian path with endvertices a and b such that $d(a,G) + d(b,G) \geq n+1$, then G is Hamiltionian.

Lemma 3.1:

If there is a partition of G into two paths with endvertices (a_1, b_1) and $(a_2, b_2), a_i \in A, b_i \in B$ such that,

$$d(a_1,G) + d(b_2,G) \geq n+1,$$
$$d(a_2,G) + d(b_1,G) \geq n+1,$$

then G is Hamiltionian.

Lemma 4.1:

If Γ is a path (a cycle) in G with $2p$ vertices and if (a, b) is an edge of G with no vertex in Γ, such that $d(a,\Gamma) + d(b,\Gamma) \geq p+1$; then the subgraph $\Gamma + (a,b)$ is traceable (Hamiltionian).

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Lemma5.1:

If Γ is a path (a cycle) with 2*p* vertices and if $a \in A$, $b \in B$ are two vertices not in Γ, such that $d(a,\Gamma) + d(b,\Gamma) \geq p + 2$, then the subgraph $\Gamma + (a,b)$ is traceable (Hamiltionian).

Lemma6.1:

If $a \in A, b \in B$ are vertices of a cycle Γ with 2*p* vertices; such that;

$$d(a^+,\Gamma) + d(b^+,\Gamma) \geq p + 2,$$

Γ contains a path *P* with endvertices *a* and *b*, such $V(\Gamma) = V(P)$.

Structure Lemma2.1:

Structure lemma: if $n = n_1 + n_2$ and for v any ,v $x \in A, y \in B,$ $d(x,G) + d(y,G) \geq n + 2$, there is a partition of *G* into two balanced bipartite subgraphs (G_1, G_2) or (Γ_1, Γ_2) such that one of the following conditions is satisfied:

1. $|V(G_i)| = 2n_i$ and if $x \in A, y \in B$ are in G_i, $d(x,G_i) + d(y,G_i) \geq n_i + 2$.

2. $|\Gamma_1| = 2(n_i - 1), \Gamma_1$ is traceable, $|\Gamma_2| = 2(n_j + 1), j \neq i, \Gamma_2$ is Hamiltionian and if $u \in A, v \in B$ are in Γ_2 $d(u,\Gamma_1) + d(v,\Gamma_2) \geq n_j + 2$,

To prove the theorem, we need to know the structure of Γ_2 when Γ_2-(x_0, y_0) is not Hamiltionian.

Structure of Γ_2 when Γ_2-(x_0, y_0) is not Hamiltonian:

Case A. there is $k, 2 \le k \le n_2 - 1$, such that the edges (x_1, y_{k+1}) and (x_k, y_{n_2}) exist.

Lemma1.2:

If $\Gamma_2 - (x_0, y_0)$ is not Hamiltionian and if the edges (x_1, y_{k+1}) and (x_k, y_{n2}) exist, then x_0 is adjacent to y_k and y_0 is adjacent to x_{k+1} and one of the subgraphs $\Gamma_2 - (x_0, y_2)$ or $\Gamma_2 - (x_1, y_0)$ is Hamiltonian. .

Case B. x_1 is adjacent to $y_1, y_2, ...y_p$ and y_{n_2} is adjacent to $x_{p+1}, ..., x_{n_2}$ for $1 \le p \le n_2$.

Lemma2.2:

If $\Gamma_2 - (x_0, y_0)$ is not Hamiltionian and if x_1 is adjacent to $y_1, ..., y_p$ and y_{n_2} is adjacent to $x_{p+1}, ..., x_{n_2}$ then

(i) $\{x_1, x_2, ..., x_p, y_{p+1}, y_{p+1}, ..., y_{n_2}\}$, is independent set.

(ii) The subgraphs $(x_1, ..., x_p), (y_0, y_1, ..., y_p)$

$$((y_{p+1}, ..., y_{n_2}), (x_{p+1}, ..., x_{n_2}, x_0))$$

Are complete bipartite subgraphs.

Lemma 3.2:

In case B, if $\Gamma_2 - (x_1, y_0)$ is not Hamiltionian, if n_2 is odd, Γ_2 is the graph E_1 with $p = (n_2 + 1)/2$, if n_2 is even, Γ_2 is the graph E_2 with $p=(n_2/2)$. For $1 \le i \le p-1$ and $p+1 \le j \le n_2$, the subgraphs $\Gamma_2 - (x_0, y_j)$ and $\Gamma_2 - (x_1, y_i)$ are Hamiltonian.

Proof of the theorem: First Case:

There are two adjacent vertices of Γ_2, adjacent to the endvertices a and b Hamiltonian path of Γ_1.

Let $x \in A, y \in B$ be adjacent vertices of Γ_2 adjacent to b and a. on a cycle of Γ_2 we consider x^+, x^-, y^+, y^-.

If x^+, x^-, y^+, y^- are not adjacent to Γ_1;

$d(x^+, \Gamma_2) + d(y^+, \Gamma_2) \ge n_1 + n_2 + 2,$

$d(x^-, \Gamma_2) + d(y^-, \Gamma_2) \ge n_1 + n_2 + 2$, since we have

$d(x^-, G) + d(y^-, G) \ge n_1 + n_2 + 2 \,\&\, d(x^+, G) + d(y^+, G) \ge n_1 + n_2 + 2.$

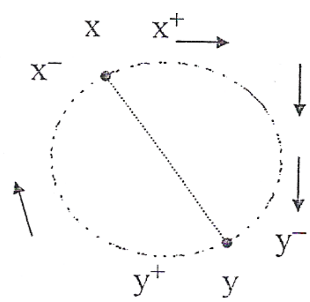
We have Γ_2 in Hamiltonian and $xy \in E$; and

$$d(x^+, \Gamma_2 - (x,y)) + d(y^+, \Gamma_2 - (x,y)) \ge n_1 + n_2 > n_2 + 1, n_1 \ge 2$$
$$d(x^-, \Gamma_2 - (x,y)) + d(y^-, \Gamma_2 - (x,y)) \ge n_1 + n_2 > n_2 + 1, n_1 \ge 2$$

Hence, there is a partition of $\Gamma_2 - (x,y)$ into two paths with endvertices (y^+, x^-) and (y^-, x^+). (See sceam. 12). Thus by Lemma, we have $\Gamma_2 - (x,y)$ is Hamiltonian, and $\Gamma_1 + (x,y)$ and $\Gamma_2 - (x,y)$ are solutions of the problem.

Else, let $(x_0 y_0 x_1 y_1 \ldots x_{n_2} y_{n_2})$ be a Hamiltonian cycle of Γ_2 such that the chardinality of the pairs of consecutive adjacent to a and b is minimum and suppose (x_0, y_0) be adjacent to b and a. If $\Gamma_2 - (x_0, y_0)$ is Hamiltonian, $\Gamma_1 + (x_0, y_0)$ and $\Gamma_2 - (x_0, y_0)$ are the solutions of the problem.

Suppose $\Gamma_2 - (x_0, y_0)$ is not Hamiltonian. We consider case A and Case B the presented paragraph.



sceam.10

Case A:

By Lemma x_0 is adjacent to y_k and y_0 is adjacent to y_{k+1}, we consider the Hamiltonian cycle of $\Gamma_2 : (y_0 x_1 \ldots x_k y_k x_0 y_{n_2} \ldots x_{k+1} y_0)$. By hypothesis of minimality, one of edges at least (a, y_k) or (b, x_{k+1}) exists, and $\Gamma_1 + (x_0, y_k)$ and $\Gamma_2 - (x_0, y_k)$ or $\Gamma_1 + (y_0, x_{k+1})$ and $\Gamma_2 + (y_0, x_{k+1})$ are the solutions of the problem; since $\Gamma_1 + (x_0, y_k)$ has a Hamiltonian cycle say; $y_k a \ldots b x_0 y_k$, also, $\Gamma_2 - (x_0, y_k)$ has a Hamiltonian cycle;

$$y_0 x_1 y_1 \ldots x_k y_{n2} x_{n2} \ldots y_{k+1} x_{k+1} x_0$$

note that in case (x_1,y_{k+1}) & (y_{n2}, x_k) are edges in Γ_2. Similarly; if the other case hold.

Case B. Subcase I:

The endvertices a and b of a Hamiltonian path of Γ_1 are adjacent to three consecutive vertices of Hamiltonian cycle of Γ_2.

We can suppose that a is adjacent to y_0 and b is adjacent to x_0 and x_1.

If $\Gamma_2-(x_1,y_0)$ is a Hamiltionian, $\Gamma_1+(x_1,y_0)$ and $\Gamma_2-(x_1,y_0)$ are solutions of the problem. (y_0a … bx_1y_0 is Hamiltonian cycle of $\Gamma_1+(x_1,y_0)$). Else $\Gamma_2-(x_1,y_0)$ is not Hamiltonian, by Lemma, Γ_2 is the graph E_1 if n_2 is odd, the graph E_2 if n_2 is even and the subgraphs $\Gamma_2-(x_0,y_j)$ for $p+1\le j\le n_2$ or $\Gamma_2-(x_1,y_j)$ for $1\le i\le p-1$ are Hamiltonian.

If a is not adjacent to y_i $1\le i\le p-1$, or y_j $p+1\le j\le n_2$ $d(a,\Gamma_2)\le 2$. By lemma, for $p+1\le j\le n_2$ $d(y_j,\Gamma_2)=n_2+1-p$ (Since y_j is independent with vertices $x_1,x_2,\ldots,x_p$, $p+1\le j\le n_2$, then

$$d(a,\Gamma_2)+d(y_j,\Gamma_2)\le n_2+3-p;$$

$$d(a,\Gamma_2)+d(y_j,\Gamma_1)\ge n+2-(n_2+3-p)=n_1+p-1;$$

$\Gamma_1+(x_0,y_i)$ contains a Hamiltonian path with endvertices a, y_j say (a, …, b, x_0, y_j). By previous Lemma, $\Gamma_1+(x_0,y_i)$ is Hamiltonian (since $d(a,\Gamma_1^*)+d(y_j,\Gamma_1^*)\ge n_1+p\ge n_1+1$, where $\Gamma_1^*=\Gamma_1+(x_0,y_i)$).

Hence, $\Gamma_1+(x_0,y_i)$ and $\Gamma_2-(x_0,y_i)$ are solutions of problem.

Subcase II:

The endvertices *a* and *b* of a Hamiltonian path of Γ_1 are adjacent to y_k and x_k, $k \neq 0$.

If $1 \leq k \leq p-1$ or $p+2 \leq k \leq n_2$, $\Gamma_1 + (x_k, y_k)$ and $\Gamma_2 - (x_k, y_k)$ are the solution of the problem.

If $k = p$, the edes (x_1, y_p) and (x_0, y_p) exist; the vertices x_0, y_0 and x_p are consecutive on the Hamiltonian cycle $(x_0 y_0 x_p y_{p-1} \ldots x_1 y_p x_{p+1} \ldots x_{n_2} y_{n_2} x_0)$ and we can conclude as in the first subcase.

If $k = p+1$, the edes (x_0, y_{p+1}) and (y_{n_2}, x_{p+1}) exist; as a similar argument, the vertices x_0, y_0, y_{p+1} are consecutive on the Hamiltonian cycle $(y_0 x_0 y_{p+1} \ldots x_{n_2} y_{n_2} x_{p+1} \ldots x_1)$ and we can conclude as in the first subcase.

Subcase III:

The endvertices *a* and *b* of a Hamiltonian path of Γ_1 satisfy $d(a,\Gamma_2) + d(b,\Gamma_2) = n_2 + 2$.

If $1 \leq k \leq p-1$ or $p+1 \leq k \leq n_2 - 1$, $\Gamma_1 + (x_k, y_{k+1})$ and $\Gamma_2 - (y_k, x_{k+1})$ are solution of the problem. Note that $\Gamma_2 - (y_k, x_{k+1})$ has Hamiltonian path with endvertices x_k and y_{k+1}.

If $k = p$, $p \leq n_2 - 2, p + 2 \leq j \leq n_2, d(y_j, \Gamma_2) = n_2 - p + 1$ (y_j is adjacent to $(x_{p+1}, x_{p+2}, ..., x_{n_2}, x_0)$; $d(a, \Gamma_2) = p + 1$; since a is adjacent to y_1, …, y_p, y_0, then

$$d(a,\Gamma_1) + d(y_j, \Gamma_1) \geq d(a,G) + d(y_j, G) - d(a,\Gamma_2) - d(y_j, \Gamma_2)$$

$$\geq n_1 + n_2 + 2 - (n_2 - p + 1) - (p + 1);$$

As $d(a,\Gamma_1) + d(y_j, \Gamma_1) \geq n_1$ and $|\Gamma_1| = 2(n_1 - 1)$, then there is an edge *uv* of Γ_1 with $u \in A$ and $\{av, y_j u\} \subseteq E$; see sceam.10, and we have $bx_j \in E$, for $p + 2 \leq j \leq n_2$, so $\Gamma_1 + (x_j, y_j)$ has a Hamiltionian cycle namely $a,...,uy_j x_j b,...,v,a$. Hence $\Gamma_1 + (x_j, y_j)$ is Hamiltionian. Not that $\Gamma_2 - (x_j, y_j)$ has a Hamiltionian path with envertices y_{j-1} and x_{j+1} and $\{x_{j+1} y_{n_2}, y_{j-1} x_{n_2}\} \subseteq E$.

Thus $\Gamma_1 + (x_j, y_j)$ and $\Gamma_2 - (x_j, y_j)$ are solution of the problem.

The case $p = n_2 - 1, p \geq 2$ is similar, and if $p = n_2 - 1 = 1$, so $n_2 = 2$, $p = 1$, therefore, *a* is adjacent to y_1, y_0 and *b* is adjacent to x_1, x_0, then $d(a,\Gamma_1) + d(b,\Gamma_1) \geq n_1$ so Γ_1 is Hamiltionian by Lemma.

Subcase IV:

For any Hamiltonian path of Γ_1, the endvertices α and β satisfy $d(\alpha,\Gamma_2) + d(\alpha,\Gamma_2) \geq n_2 + 1$.

Lemma3.2:

Under the hypothesis of subcase IV, Γ_1 is Hamiltonian and if $\alpha \in A$ and $\beta \in B$ are in Γ_1, then

$$d(\alpha,\Gamma_1)+d(\beta,\Gamma_1)\geq n_1+1,$$
$$d(\alpha,\Gamma_2)+d(\beta,\Gamma_2)\leq n_2+1,$$

And there is a Hamiltonian path in Γ_1 with endvertices α and β.

Proof: let α and β be the endvertices of a Hamiltonian path of Γ_1: $d(\alpha,\Gamma_1)+d(\beta,\Gamma_1)\geq n_1+1$.

By Lemma 2.1 Γ_1 is Hamiltonian, if α^+ is the successor of α, on a Hamiltonian cycle of $\Gamma_1 : d(\alpha,\Gamma_1)+d(\alpha^+,\Gamma_1)\geq n_1+1$ (α,α^+ are endvertices of a Hamiltonian path of Γ_1).

Suppose that $u\in A$ and $v\in B$ are in Γ_1, and satisfy $d(u,\Gamma_1)+d(v,\Gamma_1)\leq n_1$, then

$$d(u,\Gamma_1)+d(u^+,\Gamma_1)\geq n_1+1;$$

$$d(v,\Gamma_1)+d(v^+,\Gamma_1)\geq n_1+1;$$

Implies that, $d(u^+,\Gamma_1)+d(v^+,\Gamma_1)\geq n_1+2$. Therefore by Lemma Γ_1 contains a Hamiltonian path with endvertices *u*, *v*, that contradicts our hypothesis.

Proof of the theorem in subcase IV:

$d(x_1,\Gamma_2)+d(y_{n_2},\Gamma_2)=n_2+2$, so $d(x_1,\Gamma_1)+d(y_{n_2},\Gamma_1)\geq n_1$, thus x_1 and y_{n_2} are adjacent to Γ_1. By lemma 4.8.3 one of the subgraphs $\Gamma_2-(x_1,y_0)$ or $\Gamma_2-(x_0,y_{n_2})$ is Hamiltonian.

If $\Gamma_2-(x_1,y_0)$ is a Hamiltonian. Let $\delta\in\Gamma_1$ be adjacent to x_1. By **Lemma;**

$$d(\delta^+,\Gamma_1)+d(a^+,\Gamma_1)\geq n_1+1.$$

By Lemma Γ_1 contains a Hamiltonian path with endvertices δ, a. Hence $\Gamma_1 + (x_1, y_0)$ is Hamiltonian, since $(y_0 a, \ldots, \delta x_1 y_0)$ is a Hamiltonian cycle of $\Gamma_1 + (x_1, y_0)$. Hence $\Gamma_1 + (x_1, y_0)$ and $\Gamma_2 - (x_1, y_0)$ are solutions of the problem.

Proof of the theorem: Second Case:

For any Hamiltonian path of Γ_1, its endvertices *a* and *b* are not adjacent to two adjacent vertices of Γ_2.

Lemma3.3:

Under the hypothesis of second case, if $\alpha \in A, \beta \in B$ are in Γ_1.

$$d(\alpha, \Gamma_2) \geq 2,\ d(\beta, \Gamma_2) \geq 2$$
$$d(\alpha, \Gamma_1) \geq 2,\ d(\beta, \Gamma_1) \geq n_1 + 2,$$
$$d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) \leq n_2$$

Proof: Let $u \in A$ in Γ_2 be not adjacent to Γ_1 and $b \in \beta$ in Γ_1, $d(b, \Gamma_2) \geq n + 2 - d(u, G) - d(b, \Gamma_1)$, since $d(u, G) + d(b, G) \geq n_1 + n_2 + 2$;

$$d(b, G) = d(b, \Gamma_1) + d(b, \Gamma_2);$$
$$d(u, G) = d(u, \Gamma_2)$$

u is not adjacent to Γ_1; and $d(u, G) \leq n_2 + 1, d(b, \Gamma_1) \leq n_1 - 1.$

Therefore, $d(b, \Gamma_2) \geq n_1 + n_2 + 2 - n_2 - 1 - n_1 + 1;$

$$d(b, \Gamma_2) \geq 2;$$

b is adjacent to vertices $x \in A$ in Γ_2. Then $y = x^+$ is not adjacent to Γ_1 (since any two endvertices of Γ_1 is not adjacent to two adjacent vertices of Γ_2).

By a similar argument if $a \in A$ is in Γ_1, $x^+ \in \Gamma_2$, x^+ is not adjacent to Γ_1, $d(a,\Gamma_2) \geq 2$.

Suppose there are $\alpha \in A,\ B \in \beta$, two vertices of Γ_1 that satisfy

$d(\alpha,\Gamma_1) + d(\beta,\Gamma_1) = n_1 + 1$

So $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) = n_2 + n_1 + 2 - n_1 - 1 = n_2 + 1.$

Necessarily α and β are adjacent to two adjacent vertices of Γ_2 (Since $|\Gamma_2| = 2(n_2 + 1)$ and $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) = n_2 + 1$), which contradicts our hypothesis:

i.e., $d(\alpha,\Gamma_1) + d(\beta,\Gamma_1) \geq n_1 + 2$

so $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) \leq n_2$

proof of the theorem in the second case:

Let $a \in A$ and $b \in \beta$ be two vertices of Γ_1, adjacent to $y \in B$ and $x \in A$ in Γ_2. *x* and *y* are adjacent to two vertices, consecutive on a Hamiltonian cycle of Γ_2, $y' \in B$ and $x' \in A$ (note that, $x'y' \in E$ and $d(x_j,\Gamma_2) + d(y_j,\Gamma_2) \geq n_2 + 2$.

Let Γ_1'' is obviously Hamiltonian, since $x'ya,\ldots,bxy'x'$ is Hamiltonian cycle of Γ_1''.

Let $u \in A$ and $v \in B$ be two vertices of Γ_2''. We distinguish three cases:

(i) *u* and *v* are not adjacent to Γ_1.

(ii) *u* is adjacent to Γ_1 and then *u* is not adjacent to y (since y is adjacent to Γ_1).

(iii) *u* and *v* are adjacent to Γ_1 and there are not adjacent to *y* and *x*.

In each case we can conclude that:

$$d\left(u,\Gamma_2''\right)+d\left(v,\Gamma_2''\right)\geq n_2;$$

And by Lemma1.1, Γ_2'' is Hamiltonian $\left(\left|\Gamma_2''\right|=2\left(n_2-1\right)\right)$.

Let $\left(ab\alpha_2\beta_2...\alpha_{n_1-1}\beta_{n_1-1}\right)$ be a Hamiltonian cycle of Γ_1 and; if $n_1\geq 5$, let for $3\leq i\leq n_1-2,\alpha_i=\alpha,\beta_i=\beta$ be two vertices of Γ_1 different from *a* and *b*. α and β are adjacent to Γ_2'' in $y_1\in B$ and $x_1\in A\left(x_1y_1\notin E\right)$. If x_1^+ and y_1^+ are the successors of x_1 and y_1 on Hamiltonian cycle of Γ_2''.

$$d\left(y_1^+,\Gamma_2\right)+d\left(x_1^+,\Gamma_2\right)\geq n_1+n_2+2;$$

(since x_1^+, y_1^+ are not adjacent to Γ_1), then

$$d\left(y_1^+,\Gamma_2''\right)+d\left(x_1^+,\Gamma_2''\right)\geq n_1+n_2-2,\left|\Gamma_2''\right|=2\left(n_2-1\right)\geq n_2+3$$

By Lemma 4.2.6 Γ_2'' contains a Hamiltonian path with endvertices x_1, y_1 respectively; hence $\Gamma_2''+\left(\alpha,\beta\right)$ is Hamiltionian $\left(\alpha_1y_1,...,x_1\beta_1\alpha_1\right)$ is Hamiltonian cycle of $\Gamma_2''+\left(\alpha,\beta\right)$.

Let $\alpha^-=\beta_{i-1},\beta^+=\alpha_{i+1}$. By Lemma

$$d\left(\alpha^-,\Gamma_1\right)+d\left(\beta^+,\Gamma_1\right)\geq n_1+2.$$

We can deduce that $\Gamma_1''-\left(\alpha,\beta\right)$ is Hamiltonian to illustrate this;

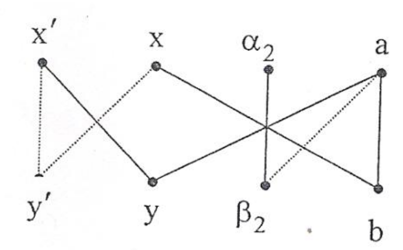
Γ_1'' is Hamiltonian i.e. $x'ya,...,bxy'x'$ is Hamiltonian cycle of Γ_1'', $\alpha = \alpha_i, \beta = \beta_i$ and $\Gamma_1'' - (\alpha,\beta)$ is Hamiltonian path with endvertices α^-, β^+ and since $d(\alpha^-, \Gamma_1) + d(\beta^+, \Gamma_1) \geq n_1 + 2$ so,

$$d(\alpha^-, \Gamma_1'' - (\alpha,\beta)) + d(\beta^+, \Gamma_1'' - (\alpha,\beta)) \geq n_1 + 1, |\Gamma_1'' - (\alpha,\beta)| = 2n_1$$

By Lemma 4.2.2 $\Gamma_1'' - (\alpha,\beta)$ is Hamiltonian. (This case satisfied if at least one of the edges $\alpha^- y$ or $\beta^+ x$ exist).

If $n_1 \geq 5, \Gamma_1'' - (\alpha,\beta)$ and $\Gamma_2'' - (\alpha,\beta)$ are solutions of the problem.

If $n_1 \leq 4$, it's easy case; to see this argument, if n_1-3, then $(ab\alpha_2\beta_2)$ is Hamiltonian cycle of Γ_1, let α_2, β_2 be adjacent to $x_1 \in A$, $y_1 \in B$ in Γ_2''. Then $\Gamma_2'' + (\alpha_2, \beta_2)$ is Hamiltonian (by the previous argument) and it's obviously $\Gamma_1'' - (\alpha_2, \beta_2)$ is Hamiltonian $(abxy'x'ya)$ is Hamiltonian cycle of $\Gamma_1'' - (\alpha_2, \beta_2)$, (See sceam.11), following the same argument for n_1=4 .



sceam.11

This completes the proof of the theorem.

References

Amar, D. (1986). Partition of a bipartite Hamiltonian graph into two cycles. *Discrete Mathematics*, *58*(1), 1–10.

Bondy, J. A., & Chvatal, V. (1976). A method in graph theory. *Discrete Mathematics*, *15*(2), 111–135.

Harary, F. (1972). Recent results on generalized Ramsey theory for graphs. In *Graph Theory and Applications* (pp. 125–138). Springer.

Harary .Frank(1994) Graph Theory.3rd Edition ,Addison-wesly, Reading, Massachusetts.

AL-Rawajfeh, A. (2012). The Cycles of simple Graphs, Mu'tah University, Karak, Jordan.

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