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On the exponentiated Weibull-Pareto distribution and properties

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Abstract

The exponentiated Weibull distribution can be used for modeling lifetime data from reliability, survival and population studies. On the other hand, Pareto distributions and their generalizations provide very flexible families of heavy-tailed distributions that may be used to model a wide variety of social and economic distributions. In this paper, we combine the above two heavy-tailed distributions, using the technique for constructing T-X family of distributions. Various structural properties have been investigated including limiting behavior, quantile, mode and kth order moment. Finally, the proposed distribution has been fitted to a real life data and the fit has been found to be good.

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1 Introduction

The exponentiated Weibull (hereafter EW in short) family, a Weibull extension obtained by adding a second shape parameter, consists of regular distributions with bathtub shaped, unimodal and a broad variety of monotone hazard rates. The EW distribution was first suggested by Mudholkar and Srivastava [1] and it has several satisfactory properties and good physical explanations, see [2] and the references therein for more details. Similarly, the Pareto distribution is well known in the literature for its capability in modeling the heavy-tailed distributions. It has been widely applied to several areas including reliability, finance and actuarial sciences.

Recently, statisticians and applied researchers are interested in constructing flexible family of distributions to model the data arising in the real world better. For example, Alzaatreh et al. [3] proposed a new method for generating family of continuous distributions. The resulting family has a connection with the hazard functions and each generated distribution is considered as a weighted hazard function of the random variable. They give several examples of the generalized family of distributions. Using the technique given in [3], Kong and Lee et al. [4] proposed the Beta-Gamma distribution and examined its related properties. Akinsete et al. [5] studied a four-parameter Beta-Pareto distribution, which has either a unimodal or a decreasing hazard rate. Explicit expressions for the mean, mean deviation, variance, skewness, kurtosis and entropies are discussed. Alzaatreh et al. [6] defined Weibull-Pareto distribution and results for moments, limiting behavior, and Shannon's entropy are provided. Alzaatreh et al. [7] further studied Gamma-Pareto (IV) distribution, various properties and some characterizations of the distribution are investigated.

Motivated by the above literature, in this article we study exponentiated Weibull-Pareto (hereafter EW-P in short) random variable. In Section 2, the EW-P distribution is defined. In Section 3, we study the distributional properties including the limiting behavior, unimodality and moments. In Section 4, we estimate the model parameters and provide the application of the EW-P distribution to real data sets in Section 5.

2 The EW-P Distribution

Let $F(x)$ be the cumulative distribution function (cdf) of any random variable X and $r(t)$ be the probability density function (pdf) of a random variable T defined on $[0, +\infty)$. The cdf of the generalized family of distributions defined by Alzaatreh et al. [3] is given by

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt, \tag{1}$$

The family of distributions defined in (1) is called 'Transformed-Transformer' (or T-X family). If a random variable T follows the EW distribution, the density function is

$$r(t) = \frac{\alpha c}{\gamma}\left(\frac{t}{\gamma}\right)^{c-1}\left[1-\exp\left\{-\left(\frac{t}{\gamma}\right)^c\right\}\right]^{\alpha-1}\exp\left\{-\left(\frac{t}{\gamma}\right)^c\right\},$$

where $t \geq 0, c > 0, \gamma > 0, \alpha > 0$, then the definition in (1) leads to the exponentiated Weibull-X family with the pdf

$$\begin{aligned} g(x) = & \frac{\alpha c}{\gamma^c} \frac{f(x)}{1 - F(x)} \left[-\log(1 - F(x)) \right]^{c-1} \exp\left\{ - \left[\frac{-\log(1 - F(x))}{\gamma} \right]^c \right\} \\ & * \left[1 - \exp\left\{ - \left[\frac{-\log(1 - F(x))}{\gamma} \right]^c \right\} \right]^{\alpha - 1}. \end{aligned} \tag{2}$$

If the random variable X follows the Pareto distribution with the pdf

$$f(x) = k\theta^k / x^{k+1}, x > \theta, \theta > 0, k > 0,$$

then (2) reduces to

$$\begin{aligned} g(x) = & \frac{\alpha c}{x} \frac{k}{\gamma} \left[-\frac{k}{\gamma} \log\left(\frac{\theta}{x} \right) \right]^{c-1} \left[1 - \exp\left\{ - \left[-\frac{k}{\gamma} \log\left(\frac{\theta}{x} \right) \right]^c \right\} \right]^{\alpha-1} \\ & * \exp\left\{ - \left[-\frac{k}{\gamma} \log\left(\frac{\theta}{x} \right) \right]^c \right\}, x > \theta. \end{aligned}$$

Let $\beta = \frac{k}{\gamma}$, the distribution in (2) can be rewritten as

$$\begin{aligned} g(x) = & \frac{\alpha c \beta}{x} \left(\beta \log\left(\frac{x}{\theta} \right) \right)^{c-1} \left[1 - \exp\left\{ - \left[\beta \log\left(\frac{x}{\theta} \right) \right]^c \right\} \right]^{\alpha-1} \\ & * \exp\left\{ - \left[\beta \log\left(\frac{x}{\theta} \right) \right]^c \right\}, x > \theta, \theta > 0, \beta > 0, c > 0, \alpha > 0. \end{aligned} \tag{3}$$

In what follows, a random variable with the pdf in (3) is said to follow the EW-P$(c, \beta, \theta, \alpha)$ distribution. After some algebras, the cdf of the EW-P distribution can be expressed as

$$G(x) = \left[1 - \exp\left\{ - \left[\beta \log\left(\frac{x}{\theta} \right) \right]^c \right\} \right]^\alpha.$$

3 Properties of the EW-P distribution

In this section, we will discuss some properties of the EW-P distribution. The following Lemma gives the relations between EW-P distribution, exponentiated Weibull distribution and exponentiated exponential distributions.

Lemma 3.1. *(a) If a random variable Y follows the exponentiated Weibull distribution with parameters c, α and $1/\beta$, then the random variable $X = \theta e^Y$ follows EW-P$(c, \beta, \theta, \alpha)$.*
(b) If a random variable Y follows the exponentiated exponential distribution with parameters 1 and α, then the random variable $X = \theta \exp\{\frac{1}{\beta} Y^{1/c}\}$ follows EW-P$(c, \beta, \theta, \alpha)$.

Proof. The results can be proved by adopting the transformation technique directly. For part (a), since since the exponentiated Weibull distribution with parameters c, α and $1/\beta$ has the pdf

$$\alpha\beta c(\beta y)^{c-1}\left[1-\exp\left\{-(\beta y)^c\right\}\right]^{\alpha-1}\exp\left\{-(\beta y)^c\right\},$$

then we have

$$\begin{aligned} P(X\le x) &= P\left(Y\le\log\left(\frac{x}{\theta}\right)\right)\\ &= \int_0^{\log\left(\frac{x}{\theta}\right)}\alpha\beta c(\beta y)^{c-1}\left[1-\exp\left\{-(\beta y)^c\right\}\right]^{\alpha-1}\exp\left\{-(\beta y)^c\right\}dy\\ &= \left[1-\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\right]^\alpha, \end{aligned}$$

which coincides with the cdf of EW-P$(c, \beta, \theta, \alpha)$.

Similarly, for part (b), since the exponentiated exponential distribution with parameters 1 and α has the pdf

$$\alpha\left(1-\exp\left(-y\right)\right)^{\alpha-1}\exp\left(-y\right),$$

we obtain

$$\begin{aligned} P(X\le x) &= P\left(Y\le\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right)\\ &= \int_0^{\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c}\alpha\left(1-\exp\left(-y\right)\right)^{\alpha-1}\exp\left(-y\right)dy\\ &= \left[1-\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\right]^\alpha, \end{aligned}$$

again, it is the cdf of random variable EW-P$(c, \beta, \theta, \alpha)$. □

For $x > \theta$, the hazard function associated with EW-P distribution can be easily calculated as

$$h_g(x)=\frac{\frac{\alpha c\beta}{x}\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c-1}\left[1-\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\right]^{\alpha-1}\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}}{1-\left[1-\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\right]^\alpha}.$$

The limiting behavior for the EW-P hazard function and the pdf are given in the following theorem.

Theorem 3.2. *The limit of the EW-P hazard functionand the pdf as $x \to +\infty$ is 0, and the limit as $x \to \theta$ is given by*

$$\lim_{x\to\theta}h_g(x)=\lim_{x\to\theta}g(x)=\begin{cases}0, & c>1\\ \alpha\beta/\theta, & c=1\\ +\infty, & c<1\end{cases}\tag{4}$$

Proof. We only give the proofs for the hazard function, the statements for the pdf can be verified similarly. According to the L'Hopital's rule, we have

$$\lim_{x\to+\infty} h_g(x) = -\lim_{x\to+\infty} \frac{\partial}{\partial x}(\log g(x)),$$

which ultimately leads to $\lim_{x\to+\infty} h_g(x) = 0$.

The formula (4) can be proved by the definition (3) and the relationship $g(x) = h_g(x)(1 - G(x))$. □

The mode of the EW-P can be obtained by solving $g'(x) = 0$. More precisely,

$$g'(x) = \frac{\alpha c \beta^2}{x^2}\left(\beta \log\left(\frac{x}{\theta}\right)\right)^{c-2}\left[1 - e^{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c}\right]^{\alpha-2} e^{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c} k(x), \quad (5)$$

where

$$\begin{aligned} k(x) = &\left(-\log\left(\frac{x}{\theta}\right) + c - 1\right)\left[1 - \exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\right] \\ &+ c\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\left[\alpha\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\} - 1\right]. \end{aligned}$$

The following theorem shows that the EW-P distribution is unimodal.

Theorem 3.3. *The EW-P distribution has a unique mode at $x = x_0$. When $c \leq 1$, the mode is $x = \theta$; when $1 < c < \frac{1}{1+\alpha}$, the mode x_0 is the solution of equation $k(x) = 0$.*

Proof. Firstly, from (5) we know that the critical points of $g(x)$ are $x = \theta$ and $x = x_0$ with $k(x_0) = 0$.

For $c \leq 1$, the inequality $0 < \exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\} < 1$ means that $g(x)$ is strictly decreasing. When $c = 1$, we have $\lim_{x\to\theta} g(x) = \frac{\alpha\beta}{\theta}$; when $c < 1$, we have $\lim_{x\to\theta} g(x) = +\infty$. Thus, $g(x)$ has a unique mode at $x = \theta$.

When $c > 1$, we have $\lim_{x\to\theta} g(x) = 0$, let $k'(x) = K_1 + K_2$ with

$$\begin{aligned} K_1 = &\frac{1}{x}\left[1 - e^{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c}\right] - \frac{\beta c}{x}\log\left(\frac{x}{\theta}\right) e^{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c}\left[\beta\log\left(\frac{x}{\theta}\right)\right]^{c-1} \\ &- \frac{\alpha\beta c^2}{x}\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{2c-1} e^{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c} - \frac{\beta c^2}{x}\left(\beta\log\left(\frac{x}{\theta}\right)\right)^{c-1}, \\ K_2 = &\frac{\beta c}{x}\exp\left\{-\left[\beta\log\left(\frac{x}{\theta}\right)\right]^c\right\}\left[\beta\log\left(\frac{x}{\theta}\right)\right]^{c-1}(c - 1 + c\alpha). \end{aligned}$$

When $c > 1, K_1 < 0$, if $c - 1 + c\alpha < 0$, we have $k'(x) < 0$, so $k(x)$ is strictly decreasing, Thus $g(x)$ has a unique mode at $x = x_0$, according to the formula $\lim_{x\to+\infty} g(x) = 0$ and $\lim_{x\to\theta} g(x) = 0$, it follows that $g(x)$ has only one mode. □

Some other related distributional propertied can also be given. For example, the quantile function of EW-P distribution is obtained by inverting $G\left(Q(\lambda)\right)=\lambda$ directly as

$$Q(\lambda)=\theta\exp\left\{\frac{1}{\beta}\left[-\log\left(1-\lambda^{1/\alpha}\right)\right]^{1/c}\right\}. \tag{6}$$

On the other hand, if X follows the EW-P distribution, the kth order moment can be calculated as

$$E\left(X^k\right)=\sum_{i=0}^{+\infty}\frac{k^i}{i!\beta^i}\sum_{n=0}^{+\infty}\frac{(\alpha-1)_n}{n!}\alpha\theta^k(n+1)^{-i/c-1}\Gamma\left(1+\frac{i}{c}\right).$$

where$(\alpha-1)_n=(\alpha-1)\cdots(\alpha-n+1),\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$.

The mean, variance, skewness, and kurtosis can be calculated from the ordinary moments using the well known relationships.

4 Parameter estimation

In this section, we will discuss the parameter estimation of the EW-P distribution. Let $X_1,X_2,\ldots,X_n$ be a random sample of size n drawn from the density in (3). The likelihood function is given by

$$\begin{aligned}L(c,\beta,\theta,\alpha)=&\prod_{i=1}^{n}\frac{\alpha c\beta}{X_i}\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^{c-1}\left[1-\exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}\right]^{\alpha-1}\\&*\exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}.\end{aligned}$$

then the log-likelihood function satisfies

$$\begin{aligned}\log L(c,\beta,\theta,\alpha)=&\,n\log\alpha+n\log c+nc\log\beta+(c-1)\sum_{i=1}^{n}\log\left(\log\left(\frac{X_i}{\theta}\right)\right)\\&-\sum_{i=1}^{n}\log X_i+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c}\right]-\sum_{i=1}^{n}\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c.\end{aligned}$$

Taking the derivatives of $\log L(c,\beta,\theta,\alpha)$ with respect to c,β,θ,α, one has

$$\begin{aligned}\frac{\partial}{\partial c}lL=&\frac{n}{c}+n\log\beta+\sum_{i=1}^{n}\frac{(\alpha-1)\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c e^{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c}\log\left(\beta\log\left(\frac{X_i}{\theta}\right)\right)}{1-e^{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c}}\\&+\sum_{i=1}^{n}\log\left(\log\left(\frac{X_i}{\theta}\right)\right)-\sum_{i=1}^{n}\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\log\left(\beta\log\left(\frac{X_i}{\theta}\right)\right),\end{aligned}\tag{7}$$

$$\frac{\partial}{\partial\beta}lL = \frac{nc}{\beta} + \sum_{i=1}^{n} \frac{c\beta^{c-1}(\alpha-1)\left[\log\left(\frac{X_i}{\theta}\right)\right]^c \exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}}{1-\exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}} - \sum_{i=1}^{n} c\beta^{c-1}\left[\log\left(\frac{X_i}{\theta}\right)\right]^c, \tag{8}$$

$$\frac{\partial}{\partial\theta}lL = -\sum_{i=1}^{n} \frac{c\beta^{c}(\alpha-1)\left[\log\left(\frac{X_i}{\theta}\right)\right]^{c-1} \exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}}{\theta\left(1-\exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}\right)} + \frac{1-c}{\theta}\sum_{i=1}^{n}\left[\log\left(\frac{X_i}{\theta}\right)\right]^{-1} + \frac{c\beta^c}{\theta}\sum_{i=1}^{n}\left[\log\left(\frac{X_i}{\theta}\right)\right]^{c-1}, \tag{9}$$

$$\frac{\partial}{\partial\alpha}lL = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-\exp\left\{-\left[\beta\log\left(\frac{X_i}{\theta}\right)\right]^c\right\}\right). \tag{10}$$

Setting (7)-(10) to zero and solving them simultaneously yields the ML estimators $\hat{c}, \hat{\beta}, \hat{\theta}, \hat{\alpha}$.

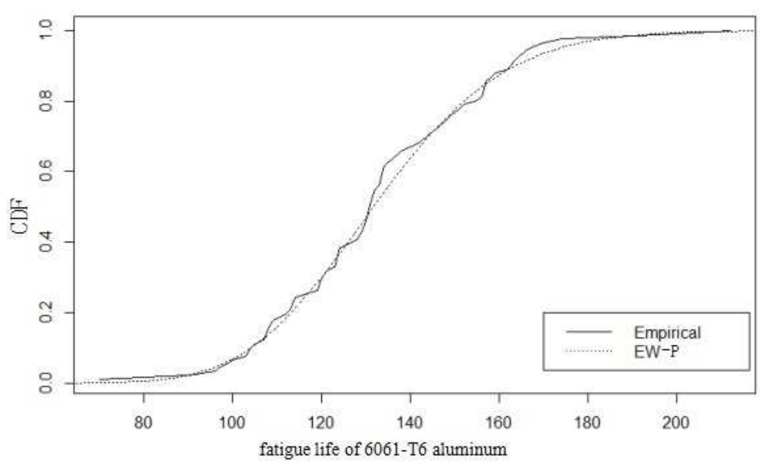

Figure 1: CDF for fitted distributions of the fatigue life of 6061-T6 Aluminium data.

5 The empirical analysis

In this section, we use the fatigue life of 6061-T6 aluminum data to present empirical analysis. The data is borrowed from the literature [7], it represents the fatigue life of 6061-T6 aluminum which is parallel with the direction of rolling and oscillated at 18 cycles per second. The maximum likelihood estimates, the log-likelihood value, the Akaike information criterion (AIC), the Kolmogorov-Smirnov (K-S) test statistic, and the p-value for the K-S statistics for five fitted distributions are reported in Table (1). The distributions are Pareto

(IV) distribution, Beta-Pareto distribution, Beta-generalized Pareto distribution, Gamma-Pareto (IV) distribution and EW-P distribution. The empirical results presented in Table (1) indicate that the T-X family of distributions provide equally adequate fit to the data, and the EW-P distribution is the most appropriated one. Figure (1) displays the empirical distribution and the fitted EW-P distribution.

Table 1: Parameter estimates for different models.

Distribution	Parameter estimates	NLL	AIC	K-S	K-S p-value
Pareto(IV)	$\hat{\beta}=0.0253$ $\hat{\gamma}=0.1234$	754.19	1512.38	0.5827	0.000
Beta-Pareto	$\hat{\alpha}=485.470$ $\hat{\beta}=162.060$ $\hat{k}=0.3943$ $\hat{\theta}=3.910$	458.65	925.30	0.091	0.376
Beta-generalized Pareto	$\hat{\alpha}=12.112$ $\hat{\beta}=1.702$ $\hat{\mu}=40.564$ $\hat{k}=0.273$ $\hat{\theta}=54.837$	457.85	925.70	0.070	0.700
Gamma-Pareto(IV)	$\hat{\alpha}=819.030$ $\hat{\gamma}=0.0637$ $\hat{c}=0.0935$	457.67	921.34	0.077	0.581
Weibull-Pareto	$\hat{\theta}=54.4686$ $\hat{\beta}=1.0522$ $\hat{c}=5.5252$	415.7771	837.5543	0.08553	0.2497
EW-P	$\hat{\alpha}=3.64483$ $\hat{\beta}=0.40276$ $\hat{\theta}=9.35514$ $\hat{c}=8.96530$	414.7583	837.5166	0.0704	0.7266

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