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On the Dynamics of a SIRS Model

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Abstract

In this paper we work in a SIRS model similar to the one given in [5], we study its dynamics under different conditions, local and global stability, and some type of bifurcation, no periodic orbits shows up. Two examples are also provided in order to illustrate particular cases of the general model, numerical implementation is performed to show behavior of the particular model in each case.

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1 Introduction

The word epidemiology means the study of epidemics and epidemic diseases, its main objective consists in research the distribution and causes of population diseases. Even when during the first decades of past century most of the efforts were made only to the study epidemic and pandemic diseases nowadays this whole picture has changed dramatically since their methods and principles are used to attack other type of diseases and health conditions as well ([4, 7, 8]).

One of the most important mathematical tools used to model real life situations are differential equations, the type of model we address here is known as SIRS (Susceptible, Infective, Recovered individuals respectively), this approach assumes the disease remains in the population for large period of time and part of the recovered populations turns back to be susceptible again, further it also suppose newborns are also susceptible.

The phenomenon we study here is describe by a set of three differential equations involving variables, depending on time, S, I, and R mentioned above.

2 Preliminary Notes

The model we consider here is inspired in [5] and given as

$$\begin{aligned} S' &= B(N) + \gamma R - bS - H(I,S)\,IS \\ I' &= H(I,S)\,IS - (b+v)\,I \\ R' &= vI - (b+\gamma)\,R. \end{aligned} \tag{1}$$

We assume that all newborn individuals are susceptible, and that all coefficients are positive; S represents susceptible individuals, I the infected and R the recovered ones; $H(I,S)$ is the incidence rate per infective individual, b is the per capita death rate, γ is the per capita rate of loss of immunity, v is the per capita recovery rate, and $B(N)$ is a C^1 function which represents the (non-negative) birth rate (a function of $N = S + I + R$). $H(I,S)$ is assumed to be differentiable; further, for all I, $H(I,0) = 0 \ \forall I$ and $\frac{\partial H}{\partial S} > 0$. The latter condition reflects the biologically intuitive requirement that the incidence rate be an increasing function of the number of susceptible.

By adding up the foregoing equations

$$S' + I' + R' = B(N) - b(S + I + R).$$

So the behavior of the population is now described by

$$N' = B(N) - bN. \tag{2}$$

Notice that solution of (2) exists locally and is unique. From now on we assume existence of $N_0 > 0$ such that $B(N_0) - bN_0 = 0$ and its asymptotically stability. In case these hypotheses do not take place we will be in the presence of extinction or uncontrollable growth of the population. In this paper we restrict our study to the case $S + I + R = N_0 > 0$.

For the sake of simplicity we normalize S, I and R, that is,

$$\overline{S} = \frac{S}{N_0}, \qquad \overline{I} = \frac{I}{N_0}, \qquad \overline{R} = \frac{R}{N_0}, \qquad b = \frac{B(N_0)}{N_0}.$$

So (1) now looks like

$$\begin{aligned} \overline{S}' &= b + \gamma\overline{R} - b\overline{S} - H(N_0\overline{I}, N_0\overline{S})N_0\overline{IS} \\ \overline{I}' &= H(N_0\overline{I}, N_0\overline{S})N_0\overline{IS} - (b+v)\overline{I} \\ \overline{R}' &= v\overline{I} - (b+\gamma)\overline{R}, \end{aligned}$$

with $\overline{S}+\overline{I}+\overline{R}=1$, and by setting $H^*(\overline{I},\overline{S}) = H(N_0\overline{I}, N_0\overline{S})N_0$, above system becomes

$$\begin{aligned} \overline{S}' &= b+\gamma\overline{R} - b\overline{S} - H^*(\overline{I},\overline{S})\overline{IS} \\ \overline{I}' &= H^*(\overline{I},\overline{S})\overline{IS} - (b+v)\overline{I} \\ \overline{R}' &= v\overline{I} - (b+\gamma)\overline{R}, \end{aligned} \tag{3}$$

omitting bars and keeping in mind we are studding (3) on $S+I+R=1$ now, it is enough to consider the first two equations that look like

$$\begin{aligned} S' &= (b+\gamma)-(b+\gamma)S - \gamma I - H^*(I,S)IS \\ I' &= H^*(I,S)IS - (b+v)I. \end{aligned} \tag{4}$$

3 Main Results

Here we state and prove the main result of this research, numerical implementation is also included.

Theorem 3.1 *Let* $\Omega = \{(I,S) : I \geq 0, S \geq 0, I+S \leq 1\}$, *then* Ω *is positive invariant for the system (4).*

Proof. Because $I'+S' = b+\gamma-(b+\gamma)(I+S) - vI \ \leq \ b+\gamma-(b+\gamma)(I+S)$, then $(I+S)' \leq (b+\gamma)(1-(I+S))$.

Since $H(I,0) > 0$ and $\frac{\partial H}{\partial S} > 0$ on Ω which is compact, so there exists $h > 0$ such that $|H(I,S)| \leq h$ on Ω.

Remark 3.2 *System (4) can be written as* $\begin{pmatrix} I' \\ S' \end{pmatrix} = f\begin{pmatrix} I \\ S \end{pmatrix}$, *with* $f\begin{pmatrix} I \\ S \end{pmatrix} = \begin{pmatrix} f_1\begin{pmatrix} I \\ S \end{pmatrix} \\ f_2\begin{pmatrix} I \\ S \end{pmatrix} \end{pmatrix}$ *hence it is readily to check existence of* $P > 0$ *such that* $||f|| \leq P$. *But then we may assure existence of global solutions of (4).*

In order to reduce number of parameters involved in (4) we perform the following transformations

$$\begin{aligned} T &= (b+v)t, \quad S(T) = S((b+v)t), \quad I(T) = I((b+v)t), \quad R(T) = R((b+v)t), \\ \sigma &= \frac{1}{b+v}, \quad \alpha = \frac{\gamma+b}{b+v}, \quad \beta = \frac{\gamma}{b+v}. \end{aligned}$$

So (4) becomes

$$\begin{aligned} S' &= \alpha - \alpha S - \beta I - \sigma H^*(I,S)IS \\ I' &= \sigma H^*(I,S)IS - I. \end{aligned} \tag{5}$$

Where the derivatives are taken with respect to the new time T, we examine the case in which incidence rate $H^*(I,S) = kI^{p-1}S^{q-1}$ with $p \geq 1$, $q > 1$, $k > 0$ ([5]), and rewrite (5) as

$$\begin{aligned} I' &= \sigma k I^p S^q - I \\ S' &= \alpha - \alpha S - \beta I - \sigma k I^p S^q. \end{aligned} \tag{6}$$

3.1 Equilibria and Stability

We now study equilibria of (6) and their stability, these are given by $E_0 = (0,1)$, $E_e^1 = (I_e, S_e^1)$, and $E_e^2 = (I_e, S_e^2)$, with $S_e^1 = \frac{1}{\sqrt[q]{\alpha k I_e^{p-1}}}$, $S_e^2 = -\frac{(1+\beta)}{\alpha} I_e + 1$ and $I_e > 0$ variable, we may say then that depending on the values of parameters there will be two positive equilibria (segment determined by S_e^2 is secant to branch of S_e^1 in the first quadrant), only one equilibrium (previous intersection is in one point) or no equilibrium (no intersection at all). We denote such a point or points by $E_e = (I_e, S_e)$.

The above situation is depicted below (figure 1) with parameters α_1=1.9, β_1=0.01, σ_1=0.7, $k_1 = 2$, $p_1 = 10$ y $q_1 = 2$, in this case two endemic (both component of point are positive) equilibria show up.

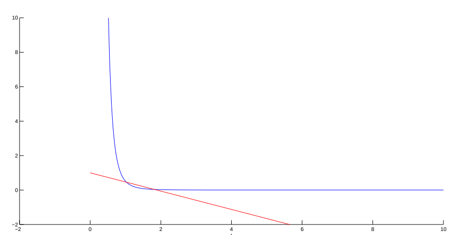

Figure 1: S_e^1 and S_e^2

The segment represents graphic of S_e^2 and the hyperbola S_e^1, if we increase p and keep fixed the rest of parameters the curve given by S_e^1 shows a similar behavior to the one in the figure concerning the number of equilibria. In conclusion, the possibilities for the equilibria are

1. Non endemic equilibria (E_0 and no intersection of curves).
2. A non endemic equilibrium and one endemic (E_0 and curves are tangent).
3. A non endemic equilibrium and two endemic (E_0 and curves are secant).

Lemma 3.3 *a) E_0 is locally asymptotically stable.*

b) E_e is locally asymptotically stable if

$$p < \min\left\{1 + \frac{(\beta+1)}{\alpha} q \frac{I_e}{S_e}, 1 + \alpha + q\frac{I_e}{S_e}\right\}.$$

Proof. Notice that Jacobian associated to (6) at (S, I) is given by

$$J(I,S) = \begin{pmatrix} \sigma kpI^{p-1}S^q - 1 & \sigma kqI^pS^{q-1} \\ -\beta - \sigma kpI^{p-1}S^q & -\alpha - \sigma kqI^pS^{q-1} \end{pmatrix} \tag{7}$$

Therefore we get the first part by a direct application of Routh-Hurwitz criterium and the last is just by checking that under the given hypothesis

$$\det\left(J\left(E_e\right)\right) > 0, \quad \operatorname{tr}\left(J\left(E_e\right)\right) < 0.$$

Theorem 3.4 *System (6) has no periodic orbits on Ω.*

Proof. Let $g(I,S) = \frac{1}{I}, I \neq 0$, then

$$\frac{\partial g f_1}{\partial I} + \frac{\partial g f_2}{\partial S} = -\alpha - \sigma kqI^pS^{q-1} < 0,$$

so by Dulac's Criterium ([3, 6]) there are no periodic orbits.

So far we have considered only $p > 1$, the case $p = 1$ it is worth to be treated separately, in this case (6) becomes

$$I' = \sigma kIS^q - IS' = \alpha - \alpha S - \beta I - \sigma kIS^q \tag{8}$$

and Jacobian

$$J\left(I,S\right) = \begin{bmatrix} \sigma kS^q - 1 & \sigma kqIS^{q-1} \\ \\ -\beta - \sigma kS^q & -\alpha - \alpha kqIS^{q-1} \end{bmatrix}. \tag{9}$$

Lemma 3.5 *a) For $\sigma k \in (0,1)$, E_0 becomes locally asymptotically stable.*

b) At $\sigma k = 1$, E_0 goes under a saddle node bifurcation.

c) For $\sigma k > 1$, E_0 is a saddle.

Proof. If λ_1, λ_2 are eigenvalues of $J(E_0)$ then

a) $\lambda_1 < 0$ and $\lambda_2 < 0$.

b) $\lambda_1 = 0$ and $\lambda_2 = -\alpha < 0$.

c) $\lambda_1 \cdot \lambda_2 < 0$.

Lemma 3.6 *If $\sigma k > 1$ then E_e is locally asymptotically stable in Ω interior.*

Theorem 3.7 *a) If $\sigma k \in (0,1)$ then E_0 is globally asymptotically stable in Ω interior.*

b) If $\sigma k > 1$ then E_0 is a saddle, its stable manifold coincides with S axis and (E_e) is globally asymptotically stable in Ω interior.

Proof.

a) Under the given hypothesis $I' < S^q - 1 < 0$ therefore $I'(t, I_0) < 0$, $t > 0$ so

$$\lim_{t \to +\infty} \psi(t, \rho) = (0, 1),$$

where $\psi(t, \rho)$ is any solution of (8) such that $\psi(0, \rho) = \rho \in R^2$.

b) It is clear E_0 is a saddle; an eigenvector corresponding to eigenvalue λ_2 (given in previous lemma) is $v = (0,1)^{\mathrm{T}}$ where T means transposed, of course the associated eigenspace is the set $\{(0,a)^{\mathrm{T}}, a \in R\}$ which coincides with S axis. The rest of the proof follows from Dulac's criterium (with $g(I,S) = \frac{1}{I}$) and Poincare-Bendixson theorem.

3.2 Examples

Finally we deal with two examples, one for [2] and the other one from [1]. The firs one is given as

$$\begin{aligned} S' &= -rSI - dS \\ I' &= rSI - aI \\ R' &= aI + dS. \end{aligned} \tag{10}$$

Because the first two equations are independent of the third one, so is enough to work with these two, that is

$$\begin{aligned} S' &= -rSI - dS \\ I' &= rSI - aI. \end{aligned} \tag{11}$$

Our first result goes as

Theorem 3.8 $R^2_+ = \{(S, I) \in R^2 : S > 0, I > 0\}$ *is positively invariant for (11) .*

Equilibria of (11) are $E_0 = (0, 0)$ and $E_1 = (\frac{a}{r}, -\frac{d}{r})$, but we disregard E_1 since it does no have sense from biological point of view. Its jacobian at E_0 is $J(E_0) = \begin{pmatrix} -d & 0 \\ 0 & -a \end{pmatrix}$, which means E_0 es locally asymptotically stable. Orbits of (11) are depicted below (Figure 2).

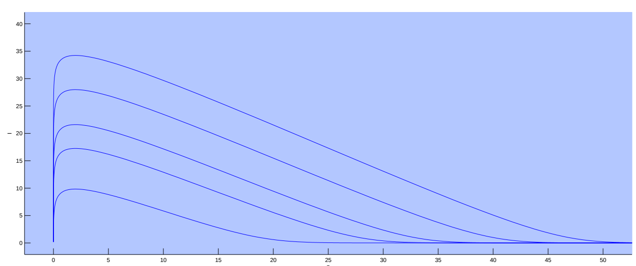

Figure 2: Susceptible vs Infected, r=0.4, d=0.6, a=0.8.

Proposition 3.9 *System (11) has no periodic orbits in* R^2_+.

Proof. Just take $g(I, S) = \frac{1}{I}$ and proceed as in pervious cases.

By using Poincare-Bendixson ([3, 6]) and foregoing proposition we assure global stability of E_0.

Theorem 3.10 *Variable* I *reaches a maximum when* $S = \frac{a}{r}$.

Proof. If we plug S into second equation of (11) then $I' = 0$ at $S = \frac{a}{r}$ and $I'' < 0$ at this value for S.

If we denote by I_{max} the maximum of I and integrate the second equation of (11) with initial data $S = S_0$ arbitrary but fixed, we get $I \equiv I(t, S_0, a) = Ke^{rS_0t}e^{-at}, t > 0$, which is increasing as a function of S_0 and a decreasing one as a function of a.

In the SIR models considered so far immunity after recovery is permanent, this is no always the case because it may decrease when time goes by, sometimes virus mutation triggers the epidemic again and in this situation immunity is not so strong. Temporal immunity can be describe by a $SIRS$ model in which transference rate from R to S is added to the model. The second model incorporates temporal immunity and is given in [1] but it has not been study in details. The model is given by

$$\begin{aligned} S' &= -\beta SI + \theta R \\ I' &= \beta SI - \alpha I \\ R' &= \alpha I - \theta R, \end{aligned} \tag{12}$$

where θ represents rate of loss of immunity and N, constant, is the whole size of the population; in this case (12) becomes

$$\begin{aligned} S' &= -\beta SI + \theta N - \theta S - \theta I \\ I' &= \beta SI - \alpha I, \end{aligned} \tag{13}$$

and as previous done, we see $\Omega = \{(S, I) : S \geq 0, I \geq 0, S + I \leq N\}$ is positively invariant for (13), the corresponding equilibria are $P_1 = (N, 0)$ and $P_2 = \left(\frac{\alpha}{\beta}, \frac{\theta}{\alpha+\theta}\left(N - \frac{\alpha}{\beta}\right)\right)$. We point out that $N\beta - \alpha < 0$ implies P_2 is out of first quadrant. The Jacobian at P_1 is $J(P_1) = \begin{pmatrix} -\theta & -(\beta N + \theta) \\ 0 & \beta N - \alpha \end{pmatrix}$, with eigenvalues $\lambda_1 = -\theta < 0$ y $\lambda_2 = \beta N - \alpha$. Notice that P_1 is a saddle if $N > \frac{\alpha}{\beta}$, locally asymptotically stable if $N < \frac{\alpha}{\beta}$ and there exists a saddle node bifurcation at $N = \frac{\alpha}{\beta}$.

In the same fashion

$$J(P_2) = \begin{pmatrix} -\left(\frac{\theta}{\alpha+\theta}\left(N\beta + \theta\right)\right) & -(\alpha + \theta) \\ \frac{\theta}{\alpha+\theta}(N\beta - \alpha) & 0 \end{pmatrix}, \det J(P_2) = \theta(N\beta - \alpha)$$

and $\mathrm{tr}J(P_2) = -\frac{\theta}{\alpha+\theta}(N\beta + \theta) < 0$. Therefore if $N > \frac{\alpha}{\beta}$ then P_2 is locally asymptotically stable and there exists a saddle node bifurcation at $N = \frac{\alpha}{\beta}$.

Remark 3.11 *Notice that either $\alpha > N\beta$ or $\alpha < N\beta$ point P_2 moves toward P_1 as $\alpha \to N\beta$.*

The incoming picture shows a bunch of orbit of (13) (Figure 3).

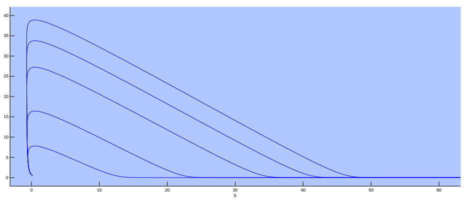

Figure 3: Susceptible vs Infected, $N = 1$, β=0.8, θ=0.6, α=0.8.

Proposition 3.12 *System (13) has no periodic orbits in* R^2_+.

Proof. Directly form Dulac's Criterium with $g(S, I) = \frac{1}{SI}$.

Now we conclude global asymptotically stability of P_1 (P_2) under the condition $N < \frac{\alpha}{\beta}$ ($N > \frac{\alpha}{\beta}$ respectively). The following picture depicts the case of a saddle-node bifurcation of (13) (Figure 4).

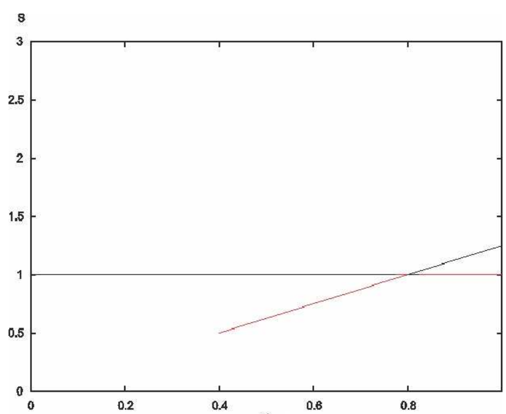

Figure 4: Saddle node bifurcation at $\alpha = 0.8 = \beta$.

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