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ON THE CONVOLUTIONS OF THE EXPONENTIAL FUNCTION AND THE EXPONENTIAL INTEGRAL

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Abstract. The exponential integral $\mathrm{ei}(\lambda x)$ and its associated functions $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ are defined as locally summable functions on the real line and their derivatives are found as distributions. In this paper the results in [1] are generalized to $x^r\,\mathrm{ei}_+(\lambda x) * x^s e_+^{\lambda x}$ and $x^r\,\mathrm{ei}_+(\lambda x) * x^s e^{\lambda x}$. Also further results are obtained.

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The *exponential integral* $\mathrm{ei}(x)$ is defined for $x > 0$ by

$$\mathrm{ei}(x) = \int_x^\infty u^{-1} e^{-u} du, \tag{1}$$

see Sneddon [9], the integral diverging for $x \le 0$. We note that equation (1) can be rewritten in the form

$$\mathrm{ei}(x) = \int_x^\infty u^{-1}[e^{-u} - H(1-u)]\, du - H(1-x)\ln|x|, \tag{2}$$

where H denotes Heaviside's function. The integral in this equation is convergent for all x and $\ln|x|$ is a locally summable function on the real line. Therefore we will use equation (2) to define $\mathrm{ei}(x)$ on the real line.

More generally, if $\lambda \ne 0$, $\mathrm{ei}(\lambda x)$ can be defined on the real line by

$$\mathrm{ei}(\lambda x) = \int_{\lambda x}^\infty u^{-1}[e^{-u} - H(1-u)]\, du - H(1-\lambda x)\ln|\lambda x|. \tag{3}$$

Further, we define $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ by

$$\mathrm{ei}_+(\lambda x) = H(x)\,\mathrm{ei}(\lambda x), \quad \mathrm{ei}_-(\lambda x) = H(-x)\,\mathrm{ei}(\lambda x)$$

so that

$$\mathrm{ei}(\lambda x) = \mathrm{ei}_+(\lambda x) + \mathrm{ei}_-(\lambda x). \tag{4}$$

In particular, if $\lambda > 0$, we have

$$\begin{aligned}
\mathrm{ei}(\lambda x) &= \int_x^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, && (5)\\
\mathrm{ei}_+(\lambda x) &= \int_x^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, \quad x > 0 &&\\
&= \int_x^\infty u^{-1}e^{-\lambda u}\,du, \quad x > 0, && (6)\\
\mathrm{ei}_-(\lambda x) &= -\gamma(\lambda) + \int_x^0 u^{-1}(e^{-\lambda u} - 1)\,du - \ln x_-, \quad x < 0, && (7)
\end{aligned}$$

where

$$\gamma(\lambda) = \gamma + \ln|\lambda|$$

and

$$\gamma = -\int_0^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du$$

is Euler's constant.

If $\lambda < 0$, we have

$$\begin{aligned}
\mathrm{ei}(\lambda x) &= -\int_{-\infty}^x u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, && (8)\\
&= \int_{-x}^\infty u^{-1}[e^{\lambda u} - H(1+\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, && (9)\\
\mathrm{ei}_+(\lambda x) &= \int_{-x}^\infty u^{-1}[e^{\lambda u} - H(1+\lambda u)]\,du - \ln|\lambda x|, \quad x > 0, &&\\
&= -\gamma(\lambda) + \int_{-x}^0 u^{-1}(e^{\lambda u} - 1)\,du - \ln x_+, \quad x > 0, &&\\
&= -\gamma(\lambda) - \int_0^x u^{-1}(e^{-\lambda u} - 1)\,du - \ln x_+, \quad x > 0, && (10)\\
\mathrm{ei}_-(\lambda x) &= \int_{-x}^\infty u^{-1}e^{\lambda u}\,du, \quad x < 0, &&\\
&= -\int_{-\infty}^x u^{-1}e^{-\lambda u}\,du, \quad x < 0. && (11)
\end{aligned}$$

The derivatives of these functions are

$$\begin{aligned}
[\mathrm{ei}(\lambda x)]' &= -e^{-\lambda x}x^{-1} = -x^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!}x^{i-1}, && (12)\\
[\mathrm{ei}_+(\lambda x)]' &= -e^{-\lambda x}x_+^{-1} - \gamma(\lambda)\delta(x) = -x_+^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!}x_+^{i-1} - \gamma(\lambda)\delta(x), && (13)\\
[\mathrm{ei}_-(\lambda x)]' &= e^{-\lambda x}x_-^{-1} + \gamma(\lambda)\delta(x) = x_-^{-1} - \sum_{i=1}^\infty \frac{\lambda^i}{i!}x_-^{i-1} + \gamma(\lambda)\delta(x). && (14)
\end{aligned}$$

Note that the following results obtained by replacing x by $-x$ in the functions $\mathrm{ei}(\lambda x)$, $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$.

$$\mathrm{ei}(\lambda(-x)) = \mathrm{ei}((-\lambda)x), \tag{15}$$
$$\mathrm{ei}_+(\lambda(-x)) = H(-x)\,\mathrm{ei}(\lambda(-x)) = \mathrm{ei}_-((-\lambda)x), \tag{16}$$
$$\mathrm{ei}_-(\lambda(-x)) = H(x)\,\mathrm{ei}(\lambda(-x)) = \mathrm{ei}_+((-\lambda)x). \tag{17}$$

These results will be used to deduce results for $\lambda < 0$ from results proved for $\lambda > 0$.

The convolution product of two functions f and g is defined as follows:

Definition 1. Let f and g be functions. Then the *convolution product* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exist.

It follows from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f. \tag{18}$$

Furthermore, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{19}$$

Gel'fand and Shilov [8] extended Definition 1 to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$, the space of infinitely differentiable functions with compact support.

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution product* $f * g$ is defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

It follows that if the convolution product $f * g$ exists by this definition then equations (18) and (19) are satisfied.

The locally summable functions $e_+^{\lambda x}$ and $e_-^{\lambda x}$ are defined for $\lambda \neq 0$ by

$$e_+^{\lambda x} = H(x)e^{\lambda x} \quad e_-^{\lambda x} = H(-x)e^{\lambda x}.$$

Note that

$$e^{\lambda(-x)} = e^{(-\lambda)x}, \quad e_+^{\lambda(-x)} = e_-^{(-\lambda)x}, \quad e_-^{\lambda(-x)} = e_+^{(-\lambda)x}. \tag{20}$$

These results will also be used to deduce results for $\lambda < 0$ from results proved for $\lambda > 0$.

The following two theorems and two corollaries were proved in [1]

Theorem 1. *The convolution $x^r \operatorname{ei}_+(x) * x^s e^x_+$ exists and*

$$\begin{aligned} x^r \operatorname{ei}_+(x) * x^s e^x_+ &= \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!}x^j e^{-x}_+ - \frac{(i-1)!}{2^i i!}e^x_+\Big] \\ &+\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}[e^x \operatorname{ei}_+(2x) - e^x \operatorname{ei}_+(x) + \ln 2\, e^x_+] \\ &-\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!} + (1-e^x)\Big]\operatorname{ei}_+(x), \end{aligned}$$

for $r, s = 0, 1, 2, \ldots$ and r, s not both zero.

In particular,

$$\begin{aligned} x^r \operatorname{ei}_+(x) * e^x_+ &= r!\sum_{i=1}^{r}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!}x^j e^{-x}_+ - \frac{(i-1)!}{2^i i!}e^x_+\Big] \\ &+r![e^x \operatorname{ei}_+(2x) - e^x \operatorname{ei}_+(x) + \ln 2\, e^x_+] \\ &-r!\Big[\sum_{i=1}^{r}\frac{x^i}{i!} + (1-e^x)\Big]\operatorname{ei}_+(x), \end{aligned}$$

for $r = 1, 2, \ldots$ and

$$\operatorname{ei}_+(x) * e^x_+ = -\operatorname{ei}_+(x) + e^x \operatorname{ei}_+(2x) + \ln 2\, e^x_+.$$

Corollary 1.1 *The convolutions $(e^{-x}x_+^{-1}) * e^x_+$ and $(e^{-x}x_+^{-2}) * e^x_+$ exist and*

$$\begin{aligned} (e^{-x}x_+^{-1}) * e^x_+ &= -e^x \operatorname{ei}_+(2x) - \gamma(2)e^x_+ \\ (e^{-x}x_+^{-2}) * e^x_+ &= 2e^x \operatorname{ei}_+(2x) + 2\gamma(2)e^x_+ - e^{-x}x_+^{-1}. \end{aligned}$$

Theorem 2. *The convolution $x^r \operatorname{ei}_+(x) * x^s e^x$ exists and*

$$\begin{aligned} x^r \operatorname{ei}_+(x) * x^s e^x &= -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{2^i i!}x^{s-k}e^x \\ &+\sum_{k=0}^{s}\binom{s}{k}(-1)^k \ln 2(r+k)!x^{s-k}e^x, \end{aligned}$$

for $r, s = 0, 1, 2, \ldots$ and r, s not both zero.

In particular

$$x^r \operatorname{ei}_+(x) * e^x = -\sum_{i=1}^{r} \frac{r!}{2^i i!} e^x + \ln 2 r! e^x,$$

for $r = 1, 2, \ldots$ *and*

$$\begin{aligned}
\operatorname{ei}_+(x) * e^x &= \ln 2\, e^x \\
\operatorname{ei}_+(x) * x e^x &= \ln 2\, x e^x - \ln 2\, e^x + \frac{1}{2} e^x.
\end{aligned}$$

Corollary 2.1 *The convolution* $(e^{-x} x_+^{-n}) * e^x$ *exists and*

$$e^{-x} x_+^{-n} * e^x = \frac{(-1)^n 2^{n-1}}{(n-1)!} \gamma(2) e^x$$

for $n = 1, 2, \ldots$.

In particular,

$$e^{-x} x_+^{-1} * x e^x = -\gamma(2) x e^x - \frac{1}{2} e^x. \tag{21}$$

To prove our results we need the following lemma which is easily proved.

Lemma.

$$\begin{aligned}
\int_0^u t^k e^{-\lambda t}\, dt &= -\sum_{i=0}^{k} \frac{k!}{i! \lambda^{k+1-i}} u^i e^{-\lambda u} + \frac{k!}{\lambda^{k+1}}, \\
\int_0^u t^k e^{-2\lambda t}\, dt &= -\sum_{i=0}^{k} \frac{k!}{(2\lambda)^{k+1-i} i!} u^i e^{-2\lambda u} + \frac{k!}{(2\lambda)^{k+1}},
\end{aligned}$$

$$\begin{aligned}
\int_0^u t^k \ln t e^{-\lambda t}\, dt &= \Big[-\sum_{i=1}^{k} \frac{k!}{i! \lambda^{k+1-i}} u^i e^{-\lambda u} + \frac{k!(1 - e^{-\lambda u})}{\lambda^{k+1}}\Big] \ln u + \\
&+ \frac{k!}{\lambda^{k+1}} \int_0^u t^{-1} (e^{-\lambda t} - 1)\, dt + \sum_{i=1}^{k} \frac{k!}{i! \lambda^{k+1-i}} \int_0^u t^{i-1} e^{-\lambda t}\, dt
\end{aligned}$$

for $k = 0, 1, 2, \ldots$.

We now prove the following generalization of Theorem 1.

Theorem 3. *If* $\lambda \neq 0$, *then the convolution* $x^r \operatorname{ei}_+(\lambda x) * x^s e_+^{\lambda x}$ *exists and*

$$\begin{aligned}
x^r \operatorname{ei}_+(\lambda x) * x^s e_+^{\lambda x} &= \sum_{k=0}^{s} \sum_{i=1}^{r+k} \binom{s}{k} \frac{(-1)^k (r+k)!}{\lambda^{r+k+1-i}} x^{s-k} \Big[\sum_{j=0}^{i-1} \frac{(i-1)!}{(2\lambda)^{i-j} i! j!} x^j e_+^{-\lambda x} - \frac{(i-1)!}{(2\lambda)^i i!} e_+^{\lambda x}\Big] \\
&+ \sum_{k=0}^{s} \binom{s}{k} \frac{(-1)^k (r+k)!}{\lambda^{r+k+1}} x^{s-k} [e^{\lambda x} \operatorname{ei}_+(2\lambda x) - e^{\lambda x} \operatorname{ei}_+(\lambda x) + \ln 2\, e_+^{\lambda x}] \\
&- \sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} \Big[\sum_{i=1}^{r+k} \frac{x^i}{i! \lambda^{r+k+1-i}} + \frac{(1 - e^{\lambda x})}{\lambda^{r+k+1}}\Big] \operatorname{ei}_+(\lambda x),
\end{aligned} \tag{22}$$

for $r, s = 0, 1, 2, \dots$ and r, s not both zero.

In particular,

$$\begin{aligned} x^r \operatorname{ei}_+(\lambda x) * e_+^{\lambda x} &= r! \sum_{i=1}^{r} \Big[\sum_{j=0}^{i-1} \frac{(i-1)!}{2^{i-j} i! j! \lambda^{r+1-j}} x^j e_+^{-\lambda x} - \frac{(i-1)!}{2^i \lambda^{r+1} i!} e_+^{\lambda x} \Big] \\ &\quad + \frac{r!}{\lambda^{r+1}} [e^{\lambda x} \operatorname{ei}_+(2\lambda x) - e^{\lambda x} \operatorname{ei}_+(\lambda x) + \ln 2\, e_+^{\lambda x}] \\ &\quad - r! \Big[\sum_{i=1}^{r} \frac{x^i}{i! \lambda^{r+1-i}} + \frac{(1-e^x)}{\lambda^{r+1}} \Big] \operatorname{ei}_+(\lambda x), \qquad (23) \end{aligned}$$

for $r = 1, 2, \dots$ and

$$\operatorname{ei}_+(\lambda x) * e_+^{\lambda x} = -\lambda^{-1} \operatorname{ei}_+(\lambda x) + \lambda^{-1} e^{\lambda x} \operatorname{ei}_+(2\lambda x) + \ln 2\, \lambda^{-1} e_+^{\lambda x}. \qquad (24)$$

Proof. First of all we prove equation (21) when $\lambda > 0$. It is obvious that $x^r \operatorname{ei}_+(\lambda x) * x^s e_+^{\lambda x} = 0$ if $x < 0$. When $x > 0$ we have

$$\begin{aligned} x^r \operatorname{ei}_+(\lambda x) * x^s e_+^{\lambda x} &= \int_0^x t^r (x-t)^s e^{\lambda(x-t)} \int_t^\infty u^{-1} e^{-\lambda u}\, du\, dt \\ &= \int_0^x u^{-1} e^{\lambda x - \lambda u} \int_0^u t^r (x-t)^s e^{-\lambda t}\, dt\, du + \int_x^\infty u^{-1} e^{\lambda x - \lambda u} \int_0^x t^r (x-t)^s e^{-\lambda t}\, dt\, du \\ &= I_1 + I_2. \qquad (25) \end{aligned}$$

Where

$$\begin{aligned} \int_0^u t^r (x-t)^s e^{-\lambda t}\, dt &= \sum_{k=0}^{s} \binom{s}{k} (-1)^k x^{s-k} \int_0^u t^{r+k} e^{-\lambda t}\, dt \\ &= -\sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} \Big[\sum_{i=1}^{r+k} \frac{u^i}{i! \lambda^{r+k+1-i}} e^{-\lambda u} + \frac{(e^{-\lambda u} - 1)}{\lambda^{r+k+1}} \Big] \qquad (26) \end{aligned}$$

and in particular for $r = s = 0$, we have

$$\int_0^u e^{-\lambda t}\, dt = -\lambda^{-1} e^{-\lambda u} + \lambda^{-1}, \qquad (27)$$

By using equation (26) we have

$$\begin{aligned} I_1 &= -\sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} e^{\lambda x} \int_0^x \sum_{i=1}^{r+k} \frac{u^{i-1}}{i! \lambda^{r+k+1-i}} e^{-2\lambda u}\, du \\ &\quad - \sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} e^{\lambda x} \int_0^x u^{-1} \frac{(e^{-2\lambda u} - e^{-\lambda u})}{\lambda^{r+k+1}}\, du. \qquad (28) \end{aligned}$$

Further by using the lemma we have

$$\int_0^x \frac{u^{i-1}}{i!\lambda^{r+k+1-i}} e^{-2\lambda u}\, du = \frac{1}{i!\lambda^{r+k+1-i}} \Big[\sum_{j=0}^{i-1} \frac{-(i-1)!}{(2\lambda)^{i-j} j!} x^j e^{-2\lambda x} + \frac{(i-1)!}{(2\lambda)^i}\Big], \quad (29)$$

and

$$\begin{aligned}\int_0^x u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\, du &= \int_0^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\, du + \\ &\quad - \int_x^\infty u^{-1} e^{-\lambda u}\, du + \int_x^\infty u^{-1} H(1-\lambda u)\, du \\ &= -\gamma - \mathrm{ei}_+(\lambda x) + \int_x^\infty u^{-1} H(1-\lambda u)\, du. \quad (30)\end{aligned}$$

Similarly

$$\int_0^x u^{-1}[e^{-2\lambda u} - H(1-2\lambda u)]\, du = -\gamma - \mathrm{ei}_+(2\lambda x) + \int_x^\infty u^{-1} H[1-2\lambda u]\, du. \quad (31)$$

On using equations (30) and (31) we have

$$\begin{aligned}\int_0^x u^{-1}(e^{-\lambda u} - e^{-2\lambda u})\, du &= \mathrm{ei}_+(2\lambda x) - \mathrm{ei}_+(\lambda x) + \int_0^\infty u^{-1}[H(1-\lambda u) - H(1-2\lambda u)]\, du \\ &= \mathrm{ei}_+(2\lambda x) - \mathrm{ei}_+(\lambda x) + \ln 2. \quad (32)\end{aligned}$$

It follows from equations (28), (29) and (32) that

$$\begin{aligned} I_1 &= \sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} \Big[\sum_{i=1}^{r+k} \frac{(i-1)!}{i!\lambda^{r+k+1-i}} \Big(\sum_{j=0}^{i-1} \frac{1}{(2\lambda)^{i-j} j!} x^j e^{-\lambda x} - \frac{e^{\lambda x}}{(2\lambda)^i}\Big)\Big] \\ &+ \sum_{k=0}^{s} \binom{s}{k} (-1)^k \lambda^{-(r+k+1)} (r+k)! x^{s-k} e^{\lambda x} \Big(\mathrm{ei}_+(2\lambda x) - \mathrm{ei}_+(\lambda x) + \ln 2\Big). \quad (33)\end{aligned}$$

In particular, for $r = s = 0$, we have

$$I_1 = \lambda^{-1}[\mathrm{ei}_+(2\lambda x) - \mathrm{ei}_+(\lambda x) + \ln 2] e_+^{\lambda x}. \quad (34)$$

Again by using equation (26), we have

$$\begin{aligned}\int_0^x t^r (x-t)^s e^{-\lambda t}\, dt &= -\sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} \Big[\sum_{i=1}^{r+k} \frac{x^i}{i!\lambda^{r+k+1-i}} e^{-\lambda x} + \\ &+ \frac{(e^{-\lambda x} - 1)}{\lambda^{r+k+1}}\Big], \quad (35)\end{aligned}$$

and so

$$I_2 = -\sum_{k=0}^{s} \binom{s}{k} (-1)^k (r+k)! x^{s-k} \Big[\sum_{i=1}^{r+k} \frac{x^i}{i!\lambda^{r+k+1-i}} + \frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big] \mathrm{ei}_+(\lambda x). \quad (36)$$

In particular, for $r = s = 0$, we have

$$I_2 = \lambda^{-1}(e^{\lambda x} - 1)\int_x^\infty u^{-1}e^{-\lambda u}\,du = \lambda^{-1}(e^{\lambda x} - 1)\,\mathrm{ei}_+(\lambda x). \tag{37}$$

Equation (24) now follows from equations (34), (37) for the case $\lambda > 0$.

Now suppose that $\lambda < 0$. Then again $x^r\,\mathrm{ei}_+(\lambda x) * x^s e_+^{\lambda x} = 0$ if $x < 0$. When $x > 0$ we have

$$\begin{aligned} x^r\,\mathrm{ei}_+(\lambda x) * x^s e_+^{\lambda x} &= -\gamma(\lambda)\int_0^x t^r(x-t)^s e^{\lambda(x-t)}\,dt - \int_0^x t^r(x-t)^s e^{\lambda(x-t)}\int_0^t u^{-1}(e^{-\lambda u} - 1)\,du\,dt \\ &- \int_0^x t^r \ln t(x-t)^s e^{\lambda(x-t)}\,dt \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{38}$$

On using the lemma we have,

$$\begin{aligned} I_1 &= -\gamma(\lambda)\sum_{k=0}^s \binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^x t^{r+k}e^{-\lambda t}\,dt \\ &= \gamma(\lambda)\sum_{k=0}^s \binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!\lambda^{r+k+1-i}} + \frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big]. \end{aligned} \tag{39}$$

In particular, for $r = s = 0$, we have

$$I_1 = \lambda^{-1}\gamma(\lambda)(1 - e^{\lambda x}). \tag{40}$$

Next we have

$$\begin{aligned} I_2 &= -\int_0^x u^{-1}(e^{-\lambda u} - 1)\int_u^x t^r(x-t)^s e^{\lambda(x-t)}\,dt\,du \\ &= -\sum_{k=0}^s \binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^x u^{-1}(e^{-\lambda u} - 1)\int_u^x t^{r+k}e^{-\lambda t}\,dt\,du \\ &= -\sum_{k=0}^s \binom{s}{k}(-1)^k(r+k)!x^{s-k}e^{\lambda x}\int_0^x u^{-1}(e^{-\lambda u} - 1)\Big[-\sum_{i=1}^{r+k}\frac{x^i e^{-\lambda x}}{i!\lambda^{r+k+1-i}} + \frac{(1-e^{-\lambda x})}{\lambda^{r+k+1}} + \\ &+ \sum_{i=1}^{r+k}\frac{u^i e^{-\lambda u}}{i!\lambda^{r+k+1-i}} + \frac{(e^{-\lambda u} - 1)}{\lambda^{r+k+1}}\Big]\,du \\ &= -\sum_{k=0}^s \binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\Big(\sum_{i=1}^{r+k}\frac{x^i}{i!\lambda^{r+k+1-i}} + \frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big) \\ &\times \Big(-\int_0^x u^{-1}(e^{-\lambda u} - 1)\,du\Big) + \frac{e^{\lambda x}}{\lambda^{r+k+1}}\int_0^x u^{-1}(e^{-2\lambda u} - e^{-\lambda u})\,du + \\ &+ \sum_{i=1}^{r+k}\frac{e^{\lambda x}}{i!\lambda^{r+k+1-i}}\int_0^x u^{i-1}(e^{-2\lambda u} - e^{-\lambda u})\,du - \frac{e^{\lambda x}}{\lambda^{r+k+1}}\int_0^x u^{-1}(e^{-\lambda u} - 1)\,du\Big]. \end{aligned} \tag{41}$$

Using equation (10) we have

$$\begin{aligned}
\int_0^x u^{-1}(e^{-2\lambda u}-e^{-\lambda u})\,du &= \int_0^x u^{-1}(e^{-2\lambda u}-1)\,du-\int_0^x u^{-1}(e^{-\lambda u}-1)\,du \\
&= -\operatorname{ei}_+(2\lambda x)+\operatorname{ei}_+(\lambda x_-\ln|2\lambda|+\ln 2 \\
&= - \quad -\operatorname{ei}_+(2\lambda x)+\operatorname{ei}_+(\lambda x)-\ln 2. \qquad (42)
\end{aligned}$$

Further we have

$$\int_0^x u^{i-1}(e^{-2\lambda u}-e^{-\lambda u})\,du = -\sum_{j=0}^{i-1}\frac{(i-1)!}{(2\lambda)^{i-j}j!}x^j e^{-2\lambda x}+\frac{(i-1)!}{(2\lambda)^i}-\int_0^x u^{i-1}e^{-\lambda u}\,du.(43)$$

Now by using equations (42) and (43) we have

$$\begin{aligned}
I_2 &= -\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\Big(\sum_{i=1}^{r+k}\frac{x^i}{i!\lambda^{r+k+1-i}}+\frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big)\Big(-\int_0^x u^{-1}(e^{-\lambda u}-1)\,du\Big)+ \\
&+ \sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\frac{e^{\lambda x}}{\lambda^{r+k+1}}\Big(\operatorname{ei}_+(2\lambda x)-\operatorname{ei}_+(\lambda x)+\ln 2\Big)+ \\
&+ \sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\sum_{i=1}^{r+k}\frac{e^{\lambda x}}{i!\lambda^{r+k+1-i}}\Big(\sum_{j=0}^{i-1}\frac{(i-1)!}{(2\lambda)^{i-j}j!}x^j e^{-2\lambda x}-\frac{(i-1)!}{(2\lambda)^i}+ \\
&+ \int_0^x u^{i-1}e^{-\lambda u}\,du\Big)+\sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\frac{e^{\lambda x}}{\lambda^{r+k+1}}\int_0^x u^{-1}(e^{-\lambda u}-1)\,du. \qquad (44)
\end{aligned}$$

In particular, for $r=s=0$, we have

$$\begin{aligned}
I_2 &= -\int_0^x u^{-1}(e^{-\lambda u}-1)\int_u^x e^{\lambda(x-t)}\,dt\,du \\
&= \lambda^{-1}\int_0^x u^{-1}(e^{-\lambda u}-1)\,du-\lambda^{-1}e^{\lambda x}\int_0^x u^{-1}(e^{-2\lambda u}-e^{-\lambda u})\,du \\
&= \lambda^{-1}\int_0^x u^{-1}(e^{-\lambda u}-1)\,du+\lambda^{-1}\Big(\operatorname{ei}_+(2\lambda x)-\operatorname{ei}_+(\lambda x)+\ln 2\Big), \qquad (45)
\end{aligned}$$

on using equation (42).

Finally we have

$$\begin{aligned}
I_3 &= -\sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^x t^{r+k}\ln t e^{-\lambda t}\,dt \\
&= \sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}(r+k)!\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!\lambda^{r+k+1-i}}+\frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big]\ln x+ \\
&- \sum_{k=0}^{s}\binom{s}{k}(-1)^k\lambda^{-(r+k+1)}x^{s-k}(r+k)!e^{\lambda x}\int_0^x t^{-1}(e^{-\lambda t}-1)\,dt+ \\
&- \sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}(r+k)!e^{\lambda x}\sum_{i=1}^{r+k}\frac{1}{i!\lambda^{r+k+1-i}}\int_0^x t^{i-1}e^{-\lambda t}\,dt. \qquad (46)
\end{aligned}$$

In particular, for $r = s = 0$, we have

$$\begin{aligned} I_3 &= -e^{\lambda x}\int_0^x \ln t e^{-\lambda t}\,dt \\ &= -\lambda^{-1}(e^{\lambda x}-1)\ln x - \lambda^{-1}e^{\lambda x}\int_0^x u^{-1}(e^{-\lambda t}-1)\,dt. \end{aligned} \tag{47}$$

Hence

$$\begin{aligned} x^r \operatorname{ei}_+(\lambda x) * x^s e_+^{\lambda x} &= \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!x^{s-k}}{i!\lambda^{r+k+1-i}}\Big(\sum_{j=0}^{i-1}\frac{(i-1)!}{(2\lambda)^{i-j}j!}x^j e^{-\lambda x} - \frac{(i-1)!e^{\lambda x}}{(2\lambda)^i}\Big) \\ +\sum_{k=0}^{s}\binom{s}{k}&(-1)^k\lambda^{-(r+k+1)}(r+k)!x^{s-k}\Big(e^{\lambda x}ei_+(2\lambda x) - e^{\lambda x}ei_+(\lambda x) + \ln 2e^{\lambda x}\Big) \\ &- \sum_{k=0}^{s}\binom{s}{k}(-1)^k(r+k)!x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{x^i}{i!\lambda^{r+k+1-i}} + \frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big] \\ &\quad\times\Big(-\gamma(\lambda) - \int_0^x u^{-1}(e^{-\lambda u}-1)\,du - \ln x\Big), \end{aligned} \tag{48}$$

on using equations (38), (39), (43) and (45) and lemma. Equation (22) now follows from equations (20) and (48).

Equation (23) follows on putting $s = 0$ in equation (22) and equation (24) follows on putting $r = 0$ in equation (23).

In the following corollary, the distribution x_+^{-2} is defined by $x_+^{-2} = (x_+^{-1})'$ and not as in Gel'fand and Shilov.

Corollary 3.1 *The convolutions* $(e^{-\lambda x}x_+^{-1}) * e_+^{\lambda x}$ *and* $(e^{-\lambda x}x_+^{-2}) * e_+^{\lambda x}$ *exist and*

$$(e^{-\lambda x}x_+^{-1}) * e_+^{\lambda x} = -e^{\lambda x}\operatorname{ei}_+(2\lambda x) - \gamma(2)e_+^{\lambda x} \tag{49}$$

$$(e^{-\lambda x}x_+^{-2}) * e_+^{\lambda x} = 2\lambda e^{\lambda x}\operatorname{ei}_+(2\lambda x) + 2\lambda\gamma(2)e_+^{\lambda x} - e^{-\lambda x}x_+^{-1}. \tag{50}$$

Proof. The convolution $(e^{-\lambda x}x_+^{-1})*e_+^{\lambda x}$ exists by Definition 2, since $e^{-\lambda x}x_+^{-1}$ and $e_+^{\lambda x}$ are both bounded on the left. From equation (24), we have

$$\begin{aligned} [\operatorname{ei}_+(\lambda x) * e_+^{\lambda x}]' &= -[e^{-\lambda x}x_+^{-1} + \gamma\delta(x)] * e_+^{\lambda x} \\ &= -(e^{-\lambda x}x_+^{-1}) * e_+^{\lambda x} - \gamma e_+^{\lambda x} \\ &= \operatorname{ei}_+(\lambda x) * [\lambda e_+^{\lambda x} + \delta(x)] \\ &= e^{\lambda x}\operatorname{ei}_+(2\lambda x) + \ln 2\, e_+^{\lambda x} \end{aligned}$$

and equation (49) follows.

From equations (19) and (49), we now have

$$\begin{aligned}[(e^{-\lambda x}x_+^{-1}) * e_+^{\lambda x}]' &= -(\lambda e^{-\lambda x}x_+^{-1} + e^{-\lambda x}x_+^{-2}) * e_+^{\lambda x} \\ &= \lambda e^{\lambda x}\operatorname{ei}_+(2\lambda x) + \lambda\gamma(2)e_+^{\lambda x} - (e^{-\lambda x}x_+^{-2}) * e_+^{\lambda x} \\ &= (e^{-\lambda x}x_+^{-1}) * [\lambda e_+^{\lambda x} + \delta(x)] \\ &= -\lambda e^{\lambda x}\operatorname{ei}_+(2\lambda x) - \lambda\gamma(2)e_+^{\lambda x} + e^{-\lambda x}x_+^{-1}\end{aligned}$$

and equation (50) follows.

Theorem 4. *If $\lambda \neq 0$, then the convolution $x^r \operatorname{ei}_-(\lambda x) * x^s e_-^{\lambda x}$ exists and*

$$\begin{aligned}&x^r \operatorname{ei}_-(\lambda x) * x^s e_-^{\lambda x} = \\ &= \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^{k+1}(r+k)!}{\lambda^{r+k+1-i}}x^{s-k}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{(2\lambda)^{i-j}i!j!}x^j e_-^{\lambda x} - \frac{(i-1)!}{(2\lambda)^i i!}e_-^{\lambda x}\Big] \\ &+\sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^{k+1}(r+k)!}{\lambda^{r+k+1}}x^{s-k}[e^{\lambda x}\operatorname{ei}_-(2\lambda x) - e^{\lambda x}\operatorname{ei}_-(\lambda x) + \ln 2\, e_-^{\lambda x}] \\ &-(-1)^{k+1}\sum_{k=0}^{s}\binom{s}{k}(r+k)!x^{s-k} \\ &\times\Big[\sum_{i=1}^{r+k}\frac{x^i i!}{\lambda^{r+k+1-i}} + \frac{(1-e^{\lambda x})}{\lambda^{r+k+1}}\Big]\operatorname{ei}_-(\lambda x) \qquad (51)\end{aligned}$$

for $r, s = 0, 1, 2, \ldots$ and r, s not both zero.

In particular,

$$\begin{aligned}x^r \operatorname{ei}_-(\lambda x) * e_-^{\lambda x} &= -r!\sum_{i=1}^{r}\Big[\sum_{j=0}^{i-1}\frac{(i-1)!}{2^{i-j}i!j!\lambda^{r+1-j}}x^j e_-^{-\lambda x} - \frac{(i-1)!}{2^i\lambda^{r+i}i!}e_-^{\lambda x}\Big] \\ &-r!\lambda^{-(r+1)}[e^{\lambda x}\operatorname{ei}_-(2\lambda x) - e^{\lambda x}\operatorname{ei}_-(\lambda x) + \ln 2\, e_-^{\lambda x}] \\ &+r!\Big[\sum_{i=1}^{r}\frac{x^i}{i!\lambda^{r+1-i}} + \frac{(1-e^x)}{\lambda^{r+1}}\Big]\operatorname{ei}_-(\lambda x), \qquad (52)\end{aligned}$$

for $r = 1, 2, \ldots$ and

$$\operatorname{ei}_-(\lambda x) * e_-^{\lambda x} = \lambda^{-1}\operatorname{ei}_-(\lambda x) - \lambda^{-1}e^{\lambda x}\operatorname{ei}_-(2\lambda x) - \ln 2\,\lambda^{-1}e_-^{\lambda x}. \qquad (53)$$

Proof.Replacing λ by $-\lambda$ in equation (22) we get

$$\begin{aligned}&x^r \operatorname{ei}_+(-\lambda x) * x^s e_+^{-\lambda x} = \\ &= \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^{r+1}(r+k)!}{\lambda^{r+k+1-i}}x^{s-k}\Big[\sum_{j=0}^{i-1}\frac{(-1)^j(i-1)!}{(2\lambda)^{i-j}i!j!}x^j e_+^{(-\lambda)x} - \frac{(i-1)!}{(2\lambda)^i i!}e_+^{(-\lambda)x}\Big] \\ &+\sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^{r+1}(r+k)!}{\lambda^{r+k+1}}x^{s-k}[e^{(-\lambda)x}\operatorname{ei}_+(2(-\lambda)x) - e^{(-\lambda)x}\operatorname{ei}_+((-\lambda)x) + \ln 2\, e_+^{(-\lambda)x}] \\ &-\sum_{k=0}^{s}\binom{s}{k}(r+k)!(-1)^{r+1}x^{s-k}\Big[\sum_{i=1}^{r+k}\frac{(-1)^i i!}{\lambda^{r+k+1-i}}x^i + \frac{(1-e^{(-\lambda)x})}{\lambda^{r+k+1}}\Big]\operatorname{ei}_+((-\lambda)x), \qquad (54)\end{aligned}$$

and equation (51) follows by replacing x by $-x$ in equation (54) and using equation (20). Equations (52) and (53) follow immediately.

Corollary 4.1 *The convolutions* $(e^{-\lambda x}x_-^{-1}) * e_-^{\lambda x}$ *and* $(e^{-\lambda x}x_-^{-2}) * e_-^{\lambda x}$ *exist and*

$$(e^{-\lambda x}x_-^{-1}) * e_-^{\lambda x} = -e^{\lambda x}\,\mathrm{ei}_-(2\lambda x) - \gamma(2)e_-^{\lambda x} \tag{55}$$

$$(e^{-\lambda x}x_-^{-2}) * e_-^{\lambda x} = -\lambda e^{\lambda x}\,\mathrm{ei}_-(2\lambda x) - \lambda 2\gamma(2)e_-^{\lambda x} - e^{-\lambda x}x_-^{-1}. \tag{56}$$

Proof. Equations (55) and (56) follow on replacing λ by $-\lambda$ and then x by $-x$ in equations (49) and (50) respectively.

Now we prove the following generalization of Theorem 2.

Theorem 5. *If* $\lambda \neq 0$, *then the convolution* $x^r\,\mathrm{ei}_+(\lambda x) * x^s e^{\lambda x}$ *exists and*

$$x^r\,\mathrm{ei}_+(\lambda x) * x^s e^{\lambda x} = -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{2^i i\lambda^{r+k+1}}x^{s-k}e^{\lambda x} + \sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^k\ln 2(r+k)!}{\lambda^{r+k+1}}x^{s-k}e^{\lambda x}, \tag{57}$$

for $r, s = 0, 1, 2, \ldots$ *and* r, s *not both zero.*

In particular

$$x^r\,\mathrm{ei}_+(\lambda x) * e^{\lambda x} = -\sum_{i=1}^{r}\frac{r!}{2^i i\lambda^{r+1}}e^{\lambda x} + \frac{\ln 2 r!}{\lambda^{r+1}}e^{\lambda x}, \tag{58}$$

for $r = 1, 2, \ldots$ *and*

$$\mathrm{ei}_+(\lambda x) * e^{\lambda x} = \ln 2\lambda^{-1}\,e^{\lambda x} \tag{59}$$

$$\mathrm{ei}_+(\lambda x) * xe^{\lambda x} = \lambda^{-1}\ln 2\,xe^{\lambda x} - \lambda^{-2}\ln 2\,e^{\lambda x} + \frac{1}{2\lambda^2}e^{\lambda x}. \tag{60}$$

Proof. We prove equation (57) for $\lambda > 0$

$$\begin{aligned} x^r\,\mathrm{ei}_+(\lambda x) * x^s e^{\lambda x} &= \int_0^\infty t^r(x-t)^s e^{\lambda(x-t)}\int_t^\infty u^{-1}e^{-\lambda u}\,du\,dt \\ &= \int_0^\infty u^{-1}e^{\lambda(x-u)}\int_0^u t^r(x-t)^s e^{-\lambda t}\,dt\,du \\ &= \sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}\int_0^\infty u^{-1}e^{\lambda x-\lambda u}\int_0^u t^{r+k}e^{-\lambda t}\,dt\,du \\ &= -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{\lambda^{r+k+1-i}}x^{s-k}e^{\lambda x}\int_0^\infty\frac{u^{i-1}}{i!}e^{-2\lambda u}\,du \\ &\quad -\sum_{k=0}^{s}\binom{s}{k}(-1)^k\lambda^{r+k+1}(r+k)!x^{s-k}e^{\lambda x}\int_0^\infty u^{-1}(e^{-2\lambda u}-e^{-\lambda u})\,du, \end{aligned} \tag{61}$$

on using the lemma.

Noting that

$$\begin{aligned}\int_0^\infty u^{-1}(e^{-2\lambda u}-e^{-\lambda u})\,du &= \int_0^\infty d[\ln u](e^{-2\lambda u}-e^{-\lambda u})\\ &= \int_0^\infty \ln u(2\lambda e^{-2\lambda u}-\lambda e^{-\lambda u})\,du\\ &= \Gamma'(1)-\ln(2\lambda)-\Gamma'(1)+\ln\lambda=-\ln 2, \quad (62)\end{aligned}$$

equation (57) now follows from equations (61), (62).

Now suppose that $\lambda<0$. Then we have

$$\begin{aligned}x^r\,\mathrm{ei}_+(\lambda x)*x^s e^{\lambda x} &= -\gamma(\lambda)\int_0^\infty t^r(x-t)^s e^{\lambda(x-t)}\,dt-\int_0^\infty t^r(x-t)^s e^{\lambda(x-t)}\int_0^t u^{-1}(e^{-\lambda u}-1)\,du\,dt\\ &- \int_0^\infty \ln t t^r(x-t)^s e^{\lambda(x-t)}\,dt = I_1+I_2+I_3 \qquad (63)\end{aligned}$$

$$\begin{aligned}I_1 &= -\gamma(\lambda)\sum_{k=0}^s\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^\infty t^{r+k}e^{-\lambda t}\,dt\\ &= -\gamma(\lambda)\sum_{k=0}^s\binom{s}{k}(-1)^k\lambda^{r+k+1}(r+k)!x^{s-k}e^{\lambda x} \qquad (64)\end{aligned}$$

$$\begin{aligned}I_2 &= -\int_0^\infty u^{-1}(e^{-\lambda u}-1)\int_u^\infty t^r(x-t)^s e^{\lambda(x-t)}\,dt\,du\\ &= -\sum_{k=0}^s\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^\infty u^{-1}(e^{-\lambda u}-1)\int_u^\infty t^{r+k}e^{-\lambda t}\,dt\,du\\ &= -\sum_{k=0}^s\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^\infty u^{-1}(e^{-\lambda u}-1)\Big(\int_0^\infty t^{r+k}e^{-\lambda t}\,dt-\int_0^u t^{r+k}e^{-\lambda t}\,dt\Big)du \qquad (65)\end{aligned}$$

Now

$$\int_0^\infty t^{r+k}e^{-\lambda t}\,dt-\int_0^u t^{r+k}e^{-\lambda t}\,dt = \sum_{i=1}^{r+k}\frac{(r+k)!u^i e^{-\lambda u}}{\lambda^{r+k+1-i}i!}+\frac{(r+k)!e^{-\lambda u}}{\lambda^{r+k+1}}. \qquad (66)$$

On using (66) we get

$$\begin{aligned}I_2 &= -\sum_{k=0}^s\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\Big[\sum_{i=1}^{r+k}\frac{(r+k)!}{\lambda^{r+k+1-i}i!}\int_0^\infty u^{i-1}(e^{-2\lambda u}-e^{-\lambda u)}\,du+\\ &- \frac{(r+k)!}{\lambda^{r+k+1}}\int_0^\infty u^{-1}(e^{-2\lambda u}-e^{\lambda u)}\,du\Big]\\ &= -\sum_{k=0}^s\binom{s}{k}\frac{(-1)^k(r+k)!}{\lambda^{r+k+1}}x^{s-k}e^{\lambda x}\Big[\sum_{i=1}^{r+k}\frac{1}{i!\lambda^{-i}}\Big(\frac{(i-1)!}{(2\lambda)^i}-\frac{(i-1)!}{\lambda^i}\Big)-\ln 2\Big]\end{aligned}$$

$$= -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{2^i i\lambda^{r+k+1}}x^{s-k}e^{\lambda x} + \sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^k\ln 2(r+k)!}{\lambda^{r+k+1}}x^{s-k}e^{\lambda x} + \\ + \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{i\lambda^{r+k+1}}x^{s-k}e^{\lambda x} \qquad (67)$$

Next we have

$$I_3 = -\sum_{k=0}^{s}\binom{s}{k}(-1)^k x^{s-k}e^{\lambda x}\int_0^\infty t^{r+k}\ln t e^{-\lambda t}\,dt$$

Now

$$\begin{aligned}\int_0^\infty t^{r+k}\ln t e^{-\lambda t}\,dt &= \frac{\Gamma'(r+k+1)}{\lambda^{r+k+1}} - \ln|\lambda|\int_0^\infty t^{r+k}e^{-\lambda t}\,dt \\ &= \frac{(r+k)!}{\lambda^{r+k+1}}\Big(-\gamma + \sum_{i=1}^{r+k}\frac{1}{i}\Big) - \frac{(r+k)!}{\lambda^{r+k+1}}\ln|\lambda| \\ &= -\frac{(r+k)!}{\lambda^{r+k+1}}\gamma(\lambda) + \frac{(r+k)!}{\lambda^{r+k+1}}\sum_{i=1}^{r+k}\frac{1}{i} \qquad (68)\end{aligned}$$

On using equation (68) we have

$$I_3 = -\sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(-1)^k(r+k)!}{i\lambda^{r+k+1}}x^{s-k}e^{\lambda x} + \gamma(\lambda)\sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^k(r+k)!}{\lambda^{r+k+1}}x^{s-k}e^{\lambda x} \qquad (69)$$

Equation (57) follows from (63), (64), (67) and (69).

Equation (58) follows on putting $s = 0$ in equation (57) and equation (59) follows on putting $r = 0$ in equation (58). Equation (60) follows on putting $r = 0$ and $s = 1$ in equation (57)

Corollary 5.1 *The convolution* $(e^{-\lambda x}x_+^{-n}) * e^{\lambda x}$ *exists and*

$$e^{-\lambda x}x_+^{-n} * e^{\lambda x} = \frac{(-1)^n\lambda 2^{n-1}}{(n-1)!}\gamma(2)e^{\lambda x} \qquad (70)$$

for $n = 1, 2, \ldots.$

In particular,

$$e^{-\lambda x}x_+^{-1} * xe^{\lambda x} = -\gamma(2)xe^{\lambda x} - \frac{1}{2\lambda}e^{\lambda x}. \qquad (71)$$

Proof. Differentiating equation (59), we get

$$[-e^{-\lambda x}x_+^{-1} - \gamma\delta(x)] * e^{\lambda x} = -(e^{-\lambda x}x_+^{-1}) * e^{\lambda x} - \gamma(\lambda)e^{\lambda x} = \ln 2e^{\lambda x}$$

this shows that equation (70) is true when $n = 1$.

Now suppose that equation (70) holds for some n. Then differentiating equation (70), we get

$$(-\lambda e^{-\lambda x}x_+^{-n} - ne^{-\lambda x}x_+^{-n-1}) * e^{\lambda x} \;=\; \lambda(e^{-\lambda x}x_+^{-n}) * e^{\lambda x}.$$

It follows that

$$\begin{aligned} ne^{-\lambda x}x_+^{-n-1} * e^{\lambda x} &= -2\lambda(e^{-\lambda x}x_+^{-n}) * e^{\lambda x} \\ &= \frac{(-1)^{n+1}\lambda 2^n}{(n-1)!}\gamma(2)e^{\lambda x} \end{aligned}$$

and we see that equation (70) holds for $n + 1$. Therefore equation (70) holds $n = 1, 2, \ldots$.

Differentiating equation (60), we get

$$[-e^{-\lambda x}x_+^{-1} - \gamma\delta(x)] * xe^{\lambda x} \;=\; \ln 2\, xe^{\lambda x} + \frac{1}{\lambda}e^{\lambda x}$$

and equation (71) follows.

Theorem 6 and its corollary follow as above.

Theorem 6. *If $\lambda \neq 0$, then the convolution $x^r \operatorname{ei}_-(\lambda x) * x^s e^{\lambda x}$ exists and*

$$\begin{aligned} x^r \operatorname{ei}_-(\lambda x) * x^s e^{\lambda x} &= \sum_{k=0}^{s}\sum_{i=1}^{r+k}\binom{s}{k}\frac{(r+k)!}{2^i i\lambda^{r+k+1}}x^{s-k}e^{\lambda x} \\ &\quad + \sum_{k=0}^{s}\binom{s}{k}(-1)^{k+1}\lambda^{-(r+k+1)}\ln 2(r+k)!x^{s-k}e^{\lambda x} \end{aligned} \tag{72}$$

for $r, s = 0, 1, 2, \ldots$ and r, s not both zero.

In particular

$$x^r \operatorname{ei}_-(\lambda x) * e^{\lambda x} \;=\; \sum_{i=1}^{r}\frac{r!}{2^i i\lambda^{r+1}}e^{\lambda x} - \ln 2\lambda^{-(r+1)}r!e^{\lambda x}, \tag{73}$$

for $r = 1, 2, \ldots$ and

$$\operatorname{ei}_-(\lambda x) * e^{\lambda x} \;=\; -\ln 2\lambda^{-1}\, e^{\lambda x} \tag{74}$$

$$\operatorname{ei}_-(\lambda x) * xe^{\lambda x} \;=-\; \lambda^{-1}\ln 2\, xe^{\lambda x} + \lambda^{-2}\ln 2\, e^{\lambda x} - \frac{1}{\lambda^2}e^{\lambda x}. \tag{75}$$

Corollary 6.1 *The convolution $(e^{-\lambda x}x_-^{-n}) * e^{\lambda x}$ exists and*

$$e^{-\lambda x}x_-^{-n} * e^{\lambda x} \;=\; \frac{(-1)^{n+1}2^{n-1}}{(n-1)!}\gamma(2)e^{\lambda x} \tag{76}$$

for $n = 1, 2, \ldots$.

In particular,

$$e^{-\lambda x}x_-^{-1} * xe^{\lambda x} = -\gamma(2)xe^{\lambda x} - \frac{1}{2\lambda}e^{\lambda x}. \quad (77)$$

References

[1] B. Fisher and F. Al-Sirehy, *Convolutions Involoving the Exponential Function and the Exponential Integral* , Mathmatica Moravica, vol. 19- 2 (2015), 65- 73.

[2] B. Fisher and J.D. Nicholas, *The exponential integral and the convolution*, SUT J. Math., **33**(2)(1997), 139-148.

[3] B. Fisher and J.D. Nicholas, *The neutrix convolution product of* ei(λx) *and* x^r, Math. Montisnigri, **9**(1998), 51-64.

[4] B. Fisher and J.D Nicholas, *On the exponential integral*, Hacet. Bull. Nat. Sci. Eng. Ser. B, **28**(1999), 55-64.

[5] B. Fisher and J.D Nicholas, *The exponential integral and the neutrix convolution product*, J. Nat. Sci. Math., **40**(1,2)(2000), 23-36.

[6] B. Fisher and J.D. Nicholas, *On the exponential integral and the non-commutative convolution product*, Integral Transforms Spec. Funct., **11**(4)(2001), 353-362.

[7] E. Özçağ and Ü. Gülen, *The exponential integral and the commutative neutrix convolution product*, J. Analysis, **7**(1999), 7-20.

[8] I.M. Gel'fand and G.E. Shilov, Generalized functions, Vol. I, Academic Press (1964).

[9] I.N. Sneddon, Special Functions of Mathematical Physics and Chemistry, Oliver and Boyd, (1961).