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On the Convergence of A Parallel Newton-Type Iteration Method For Algebraic Equation

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Abstract

A Newton-type iteration method for simultaneously finding all roots of a algebraic equation is presented. The convergence and convergence rate are discussed for the method. The results of efficiency analyses and numerical example are satisfactory.

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Key words: algebraic equation, Newton-type method, convergence, efficiency

1. Introduction

Consider algebraic equation of degree $n \ge 2$

$$p(x) = \prod_{i=1}^{n} (x - r_i) = 0 \tag{1.1}$$

with simple roots $r_1, r_2, \cdots, r_n$.

A parallel Newton-type method for equation (1.1) was suggested in paper [1], i.e.

$$x_i^{(k+1)} = x_i^{(k)} - [\frac{p(x_i^{(k)})}{p'(x_i^{(k)})}] / [1 - \frac{p(x_i^{(k)})}{p'(x_i^{(k)})} \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{x_i^{(k)} - x_j^{(k)}}] \tag{1.2}$$

Where $i = 1,2,\ldots,n; k = 0,1,2,\cdots$.

$x_i^{(0)}$ $(i = 1,2,\ldots,n)$ were distinct initial approximations for roots r_i $(i = 1,2,\ldots,n)$ of equation (1.1).

In paper [1], the convergence of the iterative method is discussed in a very tedious process.

In the following section 2, we will give a new convergence theorem for iterative method (1.2), and provide a new convergence analysis which is much more concise than that given in paper [1].

Finally, in Section 3, we give a numerical example and the computation results are satisfactory.

2. Convergence analysis

In this section we shall discuss the convergence and convergence rate of iterative method (1.2). The presented convergence analysis is more concise than that given in paper [1]

Let $k = 0,1,2,\cdots$ be the indices of iterations and

$$d = \min_{1\le i<j\le n} \left| r_i - r_j \right| \tag{2.1}$$

$$h_i^{(k)} = x_i^{(k)} - r_i \tag{2.2}$$

$$h^{(k)} = \max_{1\le i\le n} \left| h_i^{(k)} \right| \tag{2.3}$$

By simple calculation, process (1.2) can also be expressed as follows:

$$h_i^{(k+1)} = \frac{A_i^{(k)}}{1+A_i^{(k)}} h_i^{(k)}, \tag{2.4}$$

Where $h_i^{(k)}$ are defined by (2.2) and

$$A_i^{(k)} = \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{(r_j - x_j^{(k)})(x_i^{(k)} - r_i)}{(x_i^{(k)} - r_j)(x_i^{(k)} - x_j^{(k)})} \tag{2.5}$$

Theorem 2.1: *Suppose that initial approximations* $x_j^{(0)}\ (j=1,2,\cdots,n)$ *satisfy* $\left| x_j^{(0)} - r_j \right| < \frac{2d}{3+\sqrt{8n-7}}$ *, then the iterative process (1.2) converges to the roots* $r_i\ (i=1,2,\cdots,n)$ *of* $p(x)=0$*, and there exists a constant* c *(independent of* j *,* k *), such that* $\left| x_j^{(k+1)} - r_j \right| \le c \left| x_j^{(k)} - r_j \right|^3$ *, further the convergence order of iterative method (1.2) is at least* 3*.*

Proof : Suppose that $x_j^{(0)}\ (j=1,2,\cdots,n)$ satisfy the condition in theorem 2.1, then there exists a positive constant s (*independent of* j) such that $s > \frac{3+\sqrt{8n-7}}{2}$ and $\left| x_j^{(0)} - r_j \right| \le \frac{d}{s}\ (j=1,2,\cdots,n)$.

Hence we know that for $k=0$ and $i \ne j$

$$\left| x_i^{(0)} - r_j \right| \ge \left| r_i - r_j \right| - \left| x_i^{(0)} - r_i \right| \ge (s-1)\frac{d}{s},$$

$$\left| x_j^{(0)} - x_i^{(0)} \right| \ge \left| r_i - r_j \right| - \left| x_j^{(0)} - r_j \right| - \left| x_i^{(0)} - r_i \right| \ge (s-2)\frac{d}{s}.$$

By (2.5), it follows that

$$\left| A_i^{(0)} \right| \le \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{\left| h_i^{(0)} \right| \cdot \left| h_j^{(0)} \right| s^2}{(s-1)(s-2)d^2} \le \frac{n-1}{(s-1)(s-2)} < \frac{1}{2} \tag{2.6}$$

Let $$\lambda = \frac{n-1}{(s-1)(s-2)}, \qquad \mu = \frac{\lambda}{1-\lambda} \tag{2.7}$$

It is evident that $\mu < 1$.

Thus, from (2.4) we obtain that for all i

$$\left|h_i^{(1)}\right| \le \frac{\left|A_i^{(0)}\right|}{1-\left|A_i^{(0)}\right|}\left|h_i^{(0)}\right| \le \mu\left|h_i^{(0)}\right| \le \frac{d}{s} \tag{2.8}$$

Generally, if $\left|x_j^{(k)} - r_j\right| \le \frac{d}{s}$ $(j = 1, 2, \cdots, n)$, then we can obtain analogously that

$$\left|A_i^{(k)}\right| = \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{s^2\left|h_i^{(k)}\right|\cdot\left|h_j^{(k)}\right|}{(s-1)(s-2)d^2} \le \lambda < \frac{1}{2} \tag{2.9}$$

and $$\left|h_i^{(k+1)}\right| \le \mu\left|h_i^{(k)}\right| \le \frac{d}{s} \tag{2.10}$$

By mathematical induction we know that the estimates (2.9) and (2.10) hold for all $i = 1, 2, \cdots n$ and $k = 0, 1, 2, \cdots$

From (2.10) we have

$$\left|h_i^{(k)}\right| \le \mu^k\left|h_i^{(0)}\right| \le \left(\frac{d}{s}\right)\mu^k \quad \text{for} \quad i = 1, 2, \cdots n;\ k = 0, 1, 2, \cdots.$$

It is evident that $h_i^{(k)} \to 0$ ($k \to \infty$). That is $x_i^{(k)} \to r_i$ $(k \to \infty)$ for $i = 1, 2, \cdots n$ 。

Making use of (2.3) and (2.9), we have

$$\left|A_i^{(k)}\right| \le \frac{(n-1)s^2}{(s-1)(s-2)d^2}\left|h^{(k)}\right|^2 \le \frac{\lambda s^2}{d^2}\left|h^{(k)}\right|^2$$

Further, by (2.4) and $\left|A_i^{(k)}\right| < \frac{1}{2}$, we have $\left|h_i^{(k+1)}\right| \le \frac{2\lambda s^2}{d^2}\left|h^{(k)}\right|^3$.

Hence, there exists a constant c (independent of j, k), such that

$$\left|x_j^{(k+1)} - r_j\right| \le c\left|x_j^{(k)} - r_j\right|^3 .$$

So the convergence order of iterative method (1.2) is at least 3.

3. Numerical Example

In this section we will report on a numerical example. The computations were performed on IBM-PC using MATLAB.

Example : we consider the equation

$$p(x) = 32x^3 - 56x^2 + 24x - 3 = 0 .$$

It is the so-called *rayleigh equation* in theory of earthquake. The exact roots of the equation are $r_1 = \frac{1}{4}$, $r_2 = \frac{3-\sqrt{3}}{4}$, $r_3 = \frac{3+\sqrt{3}}{4}$. We want to find these roots by Newton-type method (1.2) and famous Newton method. In our computation we choose starting values $x_1^{(0)} = 0$, $x_2^{(0)} = 0.5$, $x_3^{(0)} = 1$, and take error $\varepsilon = 10^{-12}$.

The numerical results of process (1.2) are listed in Table 3.1, and for Newton method we only give the final results.

Table 3.1

Iterative method	Number of Iterations	Numerical Results		
		$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
Method (1.2)	1	0.200000000000	0.375000000000	1.176470588235
	2	0.243808087597	0.323805689748	1.183011463275
	3	0.249955665119	0.317035707337	1.183012701892
	4	0.249999999979	0.316987298131	1.183012701892
	5	0.250000000000	0.316987298108	1.183012701892
Newton method	8	0.250000000000	0.316987298108	1.183012701892

From Table 3.1 we see that for Newton method, after eight iterations it attain the precision; for Newton-type method (1.2), after five iterations it attain the precision. Hence, the modified Newton-type method converge faster than Newton method. The numerical results are satisfactory.

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