Mathematica Aeterna, Vol. 7, 2017, no. 3, 253 - 260

On the coefficient estimates of analytic and bi-univalent m-fold symmetric functions

Arzu Akgül

Department of Mathematics, Faculty of Arts and Science
Umuttepe Campus, Kocaeli University, Kocaeli, Turkey

Abstract

In this paper, we consider two new subclasses consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk. Also, we establish bounds for the coefficients for these subclasses and several related classes are considered and connections to earlier known results are made.

Mathematics Subject Classification: 30C45, 30C50

Keywords: Analytic functions, m-fold symmetric bi-univalent functions, Coefficient bounds.

1 Introduction and Preliminary Notes

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A consisting of the form (1) which are also univalent in U. The Koebe one-quarter theorem [6] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \quad (2)$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, see [17]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ (see [17]). For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in U, \quad m \in \mathbb{N}) \quad (3)$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [11], [15]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \qquad (z \in U, \quad m \in \mathbb{N}). \quad (4)$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (4). In fact, the functions in the class S are *one*-fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [19], is given as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[\left(m+1\right)a_{m+1}^2 - a_{2m+1}\right)\right]w^{2m+1} \quad (5) \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots . \end{aligned}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. For $m = 1$, the formula (5) coincides with the formula (2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[-\log(1-z^m)\right]^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right].$$

Lewin [10] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [14] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^{\star}(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$ respectively (see [14]). The classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [17] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [2], [7], [12], [16], [17], [18], [20]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([5], [8], [9]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\setminus\{1,2\}; \mathbb{N} = \{1,2,3,...\})$ is still an open problem. In this study, we derive non-sharp estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to the new general subclasses $S_{\Sigma_m}(\alpha,\lambda,\delta)$ and $S_{\Sigma_m}(\beta,\lambda,\delta)$ of Σ_m. Several related classes are also considered and connections to earlier known results are made. The subclass $S_{\Sigma_m}(\alpha,\lambda,\delta)$ and $S_{\Sigma_m}(\beta,\lambda,\delta)$ are defined as follows:

Definition 1.1 *A function $f \in \Sigma_m$ is said to be in the class $S_{\Sigma_m}(\alpha,\lambda,\delta)$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad \left|\arg\left((1-\lambda)\tfrac{f(z)}{z} + \lambda f'(z) + \delta z f''(z)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda,\ z \in U)$$

and

$$\left|\arg\left((1-\lambda)\tfrac{g(w)}{w} + \lambda g'(w) + \delta w g''(w)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda,\ w \in U)$$

where the function $g = f^{-1}$.

Definition 1.2 *A function $f \in \Sigma_m$ given by (4) is said to be in the class $S_{\Sigma_m}(\beta,\lambda,\delta)$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad Re\left((1-\lambda)\tfrac{f(z)}{z} + \lambda f'(z) + \delta z f''(z)\right) > \beta \quad ,0 < \alpha \leq 1, 0 \leq \beta < 1,\ \lambda \geq 1,\ z \in U \tag{6}$$

and

$$Re\left((1-\lambda)\tfrac{g(w)}{w} + \lambda g'(w) + \delta w g''(w)\right) > \beta \quad ,0 < \alpha \leq 1, 0 \leq \beta < 1,\ \lambda \geq 1,\ w \in U \tag{7}$$

where the function $g = f^{-1}$.

We have to remember here the following lemma here so as to derive our basic results:

Lemma 1.3 *[6] If $h \in \mathcal{P}$, then $|c_k| \leqq 2$ for each $k \in \mathbb{N} = \{1, 2, \ldots\}$, where $\mathcal{P}$ is the family of all functions h, analytic in U, for which*

$$\mathcal{R}(h(z)) > 0 \qquad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in U).$$

2 Main Results

We begin this section by finding the non sharp estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the class $S_{\Sigma_m}(\alpha, \lambda, \delta)$.

Theorem 2.1 *Let f given by (4) be in the class $S_{\Sigma_m}(\alpha, \lambda, \delta)$, $0 < \alpha \leq 1$. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{|(1 + 2\lambda m + 2\delta m(m+1))(m+1)\alpha + (1-\alpha)(1 + m\lambda + \delta m(m+1))^2|}} \tag{8}$$

and

$$|a_{2m+1}| \leq \frac{4\alpha^2}{(1 + m\lambda + m(m+1)\delta)^2} + \frac{2\alpha}{(1 + 2m\lambda + 2m(m+1)\delta}. \tag{9}$$

Let $f \in S_{\Sigma_m}(\alpha, \lambda, \delta)$. Then

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) = [p(z)]^\alpha \tag{10}$$

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) = [q(w)]^\alpha \tag{11}$$

where $g = f^{-1}$, p, q in P and have the forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots \tag{12}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots. \tag{13}$$

Now, equating the coefficients in (10) and (11), we get

$$(1 + m\lambda + \delta m(m+1))a_{m+1} = \alpha p_m, \tag{14}$$

$$(1 + 2\lambda m + 2\delta m(m+1))a_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \tag{15}$$

and

$$-(1+m\lambda+\delta m(m+1))a_{m+1}=\alpha q_m, \tag{16}$$

$$(1+2\lambda m+2\delta m(m+1))((m+1)a_{m+1}^2-a_{2m+1})=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{17}$$

From (14) and (16) we obtain

$$p_m=-q_m. \tag{18}$$

and

$$2(1+m\lambda+\delta m(m+1))a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{19}$$

Also from (15), (17) and (19) we have

$$(1+2\lambda m+2\delta m(m+1))a_{m+1}^2=\alpha\left(p_{2m}+q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

$$=\alpha\left(p_{2m}+q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}\tfrac{2(1+m\lambda+\delta m(m+1))^2}{\alpha^2}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2\left(p_{2m}+q_{2m}\right)}{(1+2\lambda m+2\delta m(m+1))(m+1)\alpha+(1-\alpha)(1+m\lambda+\delta m(m+1))^2}. \tag{20}$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\sqrt{\frac{4\alpha^2}{|(1+2\lambda m+2\delta m(m+1))(m+1)\alpha+(1-\alpha)(1+m\lambda+\delta m(m+1))^2|}}.$$

This is the desired bound for $|a_{m+1}|$ as asserted in (8). Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (17) from (15), we obtain

$$(1+2m\lambda+2\delta m(2m+1))(2a_{2m+1}-(m+1)a_{m+1}^2)=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{21}$$

It follows from (18) and (19) , and (21) that

$$a_{2m+1}=\frac{\alpha^2(p_m^2+q_m^2)(m+1)}{2(1+m\lambda+\delta m(m+1))^2}+\frac{\alpha\left(p_{2m}-q_{2m}\right)}{2(1+2m\lambda+2\delta m(2m+1))}. \tag{22}$$

Applying Lemma 1 for the coefficients p_{2m}, p_m and q_{2m}, q_m , we have

$$|a_{2m+1}|\leq\frac{2\alpha^2(m+1)}{(1+m\lambda+\delta m(m+1))^2}+\frac{2\alpha}{(1+2m\lambda+2\delta m(2m+1))}.$$

This completes the proof of Theorem 3.1.

Theorem 2.2 *Let f given by (4) be in the class $S_{\Sigma_m}(\beta,\lambda,\delta)$, $0<\alpha\leq 1, 0\leq\beta<1$, $\lambda\geq 1$. Then*

$$|a_{m+1}|\leq\sqrt{\frac{2(1-\beta)}{1+2m\lambda+2m(2m+1)\delta}} \tag{23}$$

$$|a_{2m+1}|\leq\frac{4\alpha^2}{(1+m\lambda+m(m+1)\delta)^2}+\frac{2\alpha}{(1+2m\lambda+2m(m+1)\delta} \tag{24}$$

Let $f\in S_{\Sigma_m}(\beta,\lambda,\delta)$, $\lambda\geq 0$ and $0\leq\beta<1$. Then we can write the argument inequalities in given by (6) and (7) and equivalently as follows:

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta zf''(z)=\beta+(1-\beta)p(z)\quad (z\in U) \tag{25}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta wg''(w)=\beta+(1-\beta)q(w)\quad (w\in U) \tag{26}$$

where $p(z)$ and $q(w)$ given by the equalities (12) and (13). Equating coefficients (25) and (26) yields

$$(1+m\lambda+m(m+1)\delta)a_{m+1}=(1-\beta)p_m, \tag{27}$$

$$(1+2m\lambda+2m(2m+1)\delta)a_{2m+1}=(1-\beta)p_{2m}, \tag{28}$$

and

$$-(1+m\lambda+m(m+1)\delta)a_{m+1}=(1-\beta)q_m \tag{29}$$

$$(1+2m\lambda+2m(2m+1)\delta)(2a_{m+1}^2-a_{2m+1})=(1-\beta)q_{2m}. \tag{30}$$

From (27) and (28) , we have

$$p_m=-q_m \tag{31}$$

and

$$2(1+m\lambda+m(m+1)\delta)^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{32}$$

Also, adding (27) to (29), we get

$$2(1+m\lambda+m(m+1)\delta)a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{33}$$

Applying Lemma 1 for equality (7), we have

$$|a_{m+1}|^2\leq\frac{2(1-\beta)}{1+2m\lambda+2m(2m+1)\delta}$$

This gives the bound on $|am+1|$ as asserted in (23). Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (28) from (30), we get

$$2(1+2m\lambda+2m(2m+1)\delta)(a_{2m+1}-a_{m+1}^2)=(1-\beta)(p_{2m}-q_{2m}) \qquad (34)$$

which, upon substitution of value of a_2^2 from (32) yields

$$a_{m+1}^2=\frac{(1-\beta)^2(p_m^2+q_m^2)}{2(1+m\lambda+m(m+1)\delta)}$$

Applying the Lemma 1 for the coefficients p_1, q_1, p_2 and q_2 , we readily get

$$|a_{2m+1}|\leq\frac{4(1-\beta)^2}{(1+m\lambda+m(m+1)\delta)}+\frac{2(1-\beta)}{1+2m\lambda+2m(2m+1)\delta}.$$

ACKNOWLEDGEMENTS. The author wish to thank the referees for their valuable suggestions which improved the presentation of the paper.

References

[1] Ş. Altınkaya and S. Yalçın, On some subclasses of m-fold symmetric bi-univalent functions, arXiv preprint arXiv:1603.01120 ,(2016).

[2] Ş. Altınkaya and S. Yalçın, Coefficient Bounds for Certain Subclasses of-Fold Symmetric Biunivalent Functions, Journal of Mathematics 2015 (2015), (2015) .

[3] D. A. Brannan and J. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York: Academic Press, (1979).

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Universitatis Babeş-Bolyai, Mathematica,31 (2) (1986) 70-77.

[5] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C. R. Acad. Sci. Paris, Ser. 352 (6) (2014) 479-484.

[6] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften Springer, New York, USA (1983), 259.

[7] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Applied Mathematics Letters, 24 (2011) 1569-1573.

[8] S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, C. R. Acad. Sci. Paris, Ser.I, 352 (1) (2014) 17–20.

[9] J. M. Jahangiri and S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, International Journal of Mathematics and Mathematical Sciences, Article ID 190560, (2013) 4 p.

[10] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967) 63-68.

[11] W. Koepf, Coefficient of symmetric functions of bounded boundary rotations, Proc. Amer. Math. Soc., 105 (1989) 324-329.

[12] N. Magesh and J. Yamini, Coefficient bounds for a certain subclass of bi-univalent functions, International Mathematical Forum 8, 27, (2013) 337-1344.

[13] G. Murugusundaramoorty, N. Magesh and V. Prameela, Coefficient bounds for certain classes of bi-univalent function, Abstract and Applied Analysis, Article ID 573017, (2013) 3 pp.

[14] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Archive for Rational Mechanics and Analysis, 32 (1969) 100-112.

[15] C. Pommerenke, Univalent FunctionsVandenhoeck & Ruprecht, Göttingen, 1975.

[16] F. M. Sakar and H. O. Guney, Faber Polynomial Coefficient Estimates for Subclasses of m-Fold Symmetric Bi-univalent Functions Defined by Fractional Derivative, Malaysian Journal Of Mathematical Sciences, 11 2, (2017), 275-287.

[17] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Applied Mathematics Letters 23 (10) , (2010) 1188-1192.

[18] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27 (5), (2013) 831-842.

[19] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient boundss for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Math. J. 7 (2), (2014) 1-10.

[20] Q. H. Xu, Y. C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Applied Mathematics Letters, 25, (2012) 990-994.

Received: August 25, 2017